

Answers numero uno:

1. Given a vector field $\underline{v}(\underline{r})$. Consider two arbitrary Cartesian axis systems $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ and $\{\hat{x}'_1, \hat{x}'_2, \hat{x}'_3\}$. We can write $\underline{r} = r_i \hat{x}_i = r'_i \hat{x}'_i$ and $\underline{v} = v_i \hat{x}_i = v'_i \hat{x}'_i$. The transformation matrix from unprimed to primed is $Q_{ij} = \hat{x}_i \cdot \hat{x}'_j = \partial r'_i / \partial r_j$, and that from primed to unprimed is $Q_{ij}^{-1} = \hat{x}'_i \cdot \hat{x}_j = \partial r_i / \partial r'_j$. The transformation is orthogonal, i.e. $Q_{ik} Q^{-1}_{kj} = Q^{-1}_{ik} Q_{kj} = \delta_{ij}$. Now consider the divergence $\underline{v} \cdot \underline{v}(\underline{r}) = \partial v_k / \partial r_k = (\partial / \partial r_k) v_k = (\partial / \partial r_k) Q^{-1}_{kj} v'_j = Q^{-1}_{kj} (\partial v'_j / \partial r_k) = Q^{-1}_{kj} (\partial v'_j / \partial r'_i) (\partial r'_i / \partial r_k) = Q^{-1}_{kj} (\partial v'_j / \partial r'_i) Q_{ik} = Q_{ik} Q^{-1}_{kj} (\partial v'_j / \partial r'_i) = \delta_{ij} (\partial v'_j / \partial r'_i) = \partial v'_j / \partial r'_j$. This establishes the invariance of $\underline{v} \cdot \underline{v}$.

2. (a). The paths of the jello particles are circles parallel to the $\hat{x}_1, \hat{x}_2$ plane and centered on the $\hat{x}_3$ axis. Particles on the $\hat{x}_3$ axis don't move. The whole ~~lump~~ lump of jello is rotating rigidly with angular velocity ω about the $\hat{x}_3$ axis (it's very strong jello). The particle velocities are $u_L^1 = -\omega x_1 \sin \omega t + \omega x_2 \cos \omega t$, $u_L^2 = -\omega x_1 \cos \omega t - \omega x_2 \sin \omega t$,

$u_L^3 = 0$, and the acceleration is
 ~~a_L^1~~ $a_L^1 = -\omega^2 x_1 \cos \omega t - \omega^2 x_2 \sin \omega t$, $a_L^2 = \omega^2 x_1 \sin \omega t - \omega^2 x_2 \cos \omega t$, $a_L^3 = 0$.

(b). $u_E^1 = \omega r_2$, $u_E^2 = -\omega r_1$, $u_E^3 = 0$.

Thus $\underline{u}_E = \omega \hat{x}_3 \times \underline{r}$, as usual for rigid body rotation.

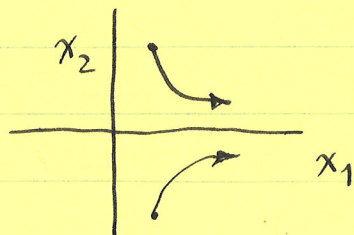
3. (a). $dr_1/dt = r_1$, $dr_2/dt = -r_2$, $dr_3/dt = 0$.

Thus $r_1 = C_1 e^t$, $r_2 = C_2 e^{-t}$, $r_3 = C_3$. At

$t=0$, $r_i = x_i$, hence $C_i = x_i$, so

the Lagrangian description of the motion is $\underline{r}(\underline{x}, t) = \hat{x}_1 x_1 e^t + \hat{x}_2 x_2 e^{-t} + \hat{x}_3 x_3$.

Since $r_1 r_2 = x_1 x_2$, the particle paths are hyperbolas in planes parallel to $\hat{x}_1 - \hat{x}_2$ plane.



(b). $\underline{x} = (1, 1, 0)$ $\rho(\underline{x}, 0) = 7$

$\underline{r}(\underline{x}, t) = (x_1 e^t, x_2 e^{-t}, x_3)$

We have, in Lagrangian form

$$\rho(\underline{x}, t) \left| \frac{\partial(\underline{r})}{\partial(\underline{x})} \right| = \rho(\underline{x}, 0) = 7$$

The Jacobian is $\frac{\partial(\underline{r})}{\partial(\underline{x})} = \begin{vmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1,$

so $\rho(\underline{x}, 8) = \rho(\underline{x}, 0) = 7.$

In Eulerian form $D_t \rho + \rho \underline{v} \cdot \underline{u} = 0$, but $\underline{v} \cdot \underline{u} = \partial u_i / \partial r_i = 0$, so $D_t \rho = 0$. The density of a particle does not change.

4. The flow must be steady, or unidirectional.

$$5. \quad \frac{d}{dt} \int_{V(t)} \rho \psi dV = \int_{V(t)} \left[\partial_t (\rho \psi) + \underline{v} \cdot (\rho \psi \underline{u}) \right] dV$$

$$= \int_{V(t)} \left\{ \rho \left[\partial_t \psi + \underline{u} \cdot \underline{v} \psi \right] + \psi \underbrace{\left[\partial_t \rho + \underline{v} \cdot (\rho \underline{u}) \right]}_{\text{zero, by continuity equation}} \right\} dV$$

$$= \int_{V(t)} \rho D_t \psi dV.$$

6. Consider a volume $V(t)$ moving with the water. The total mass of radioactive material in $V(t)$ is

$$m(t) = \int_{V(t)} \sigma dV.$$

This mass disappears due to radioactive decay at a rate $-\lambda m$, and leaves $V(t)$ through the walls $\partial V(t)$ at a rate

$$- \int_{\partial V(t)} \hat{n} \cdot \underline{F} dA.$$

The net loss due to these two effects is

$$\frac{dm}{dt} = -\lambda m - \int_{\partial V(t)} \hat{n} \cdot \underline{F} dA = -\lambda \int_{V(t)} \sigma dV - \int_{V(t)} \underline{v} \cdot \underline{F} dV,$$

By the rule for differentiating integrals over moving volumes, we may also write

$$\frac{dm}{dt} = \int_{V(t)} \partial_t \sigma dV + \int_{\partial V(t)} \hat{n} \cdot \sigma \underline{u} dA = \int_{V(t)} [\partial_t \sigma + \underline{v} \cdot (\sigma \underline{u})] dV.$$

Equating, ~~by~~ by the vanishing integral theorem, we find $\partial_t \sigma + \underline{v} \cdot (\sigma \underline{u}) = -\lambda \sigma - \underline{v} \cdot \underline{F}$ or

$$\partial_t \sigma + \underline{v} \cdot (\sigma \underline{u} + \underline{F}) = -\lambda \sigma.$$

flux due to advection $\nearrow$

$\nearrow$ loss due to radioactive decay

$\nwarrow$ flux due to concentration gradients (diffusion).

This has the expected form of a conservation law.

Stressful answers #2.

1. Denote (r_1, r_2, r_3) by (x, y, z) .

(a). The components of traction are

$$\frac{1}{a} \begin{pmatrix} x & 0 & 0 \\ 0 & z & -y \\ 0 & -y & z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{a} \begin{pmatrix} x^2 \\ 0 \\ z^2 - y^2 \end{pmatrix}. \quad \text{The traction}$$

$$\text{is thus } \underline{F} = \frac{1}{a} [\hat{x} x^2 + \hat{z} (z^2 - y^2)].$$

The torque is (per unit area)

$$\underline{r} \times \underline{F} = \frac{1}{a} [\hat{x} y (z^2 - y^2) + \hat{y} x (xz + y^2 - z^2) - \hat{z} x^2 y].$$

(b). Total force = $\int_A \underline{F} dA$. But

$$\begin{aligned} \int_A x^2 dA &= \int_A y^2 dA = \int_A z^2 dA = \frac{1}{3} \int_A (x^2 + y^2 + z^2) dA \\ &= \frac{a^2}{3} \int_A dA = \frac{4}{3} \pi a^4, \quad \text{so total force} \end{aligned}$$

$$= \frac{4}{3} \pi a^3 \hat{x}. \quad \text{Total torque} = 0$$

because the sum of $\underline{r} \times \underline{F}$ at antipodal points vanishes.

(c).

$$\int_A \underline{F} dA = \hat{x}_j \int_A n_i T_{ij} dA = \hat{x}_j \int_V \partial_i T_{ij} dV.$$

$$\text{and } (\partial_i T_{i1}, \partial_i T_{i2}, \partial_i T_{i3}) =$$

$$(\partial_1, \partial_2, \partial_3) \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = (\partial_x, \partial_y, \partial_z) \begin{pmatrix} x & 0 & 0 \\ 0 & z & -y \\ 0 & -y & z \end{pmatrix}$$

$$= (1, 0, 0), \quad \text{so } \int_A \underline{F} dA = \hat{x} \int_V dV = \frac{4}{3} \pi a^3 \hat{x}.$$

$$\text{Total torque} = \int_F \underline{r} \times \underline{F} dA = \hat{x}_i \varepsilon_{ijk} \int_A r_j F_k dA$$

$$= \hat{x}_i \varepsilon_{ijk} \int_A r_j n_l T_{lk} = \hat{x}_i \varepsilon_{ijk} \int_A n_l (r_j T_{lk})$$

$$= \hat{x}_i \varepsilon_{ijk} \int_V \partial_l (r_j T_{lk}) dV = \hat{x}_i \varepsilon_{ijk} \int_V [\delta_{lj} T_{lk} + r_j \partial_l T_{lk}] dV.$$

Now $\partial_l T_{lk} = \delta_{lk}$ (see above) and

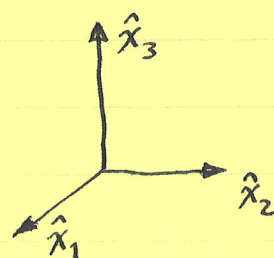
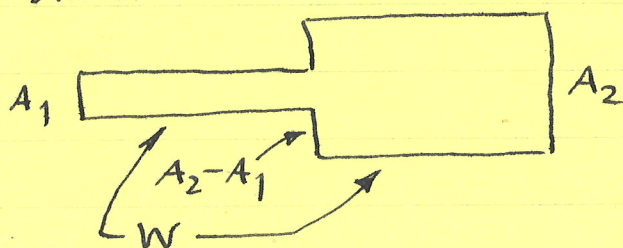
$$\varepsilon_{ijk} \delta_{lj} T_{lk} = \varepsilon_{ilk} T_{lk} = 0, \text{ since } T_{lk} = T_{kl},$$

$$\text{so total torque} = \hat{x}_i \varepsilon_{ij1} \int_V r_j dV, \text{ but}$$

$$\int_V x dV = \int_V y dV = \int_V z dV = 0, \text{ so total torque} = 0.$$

2. The downstream velocity follows from conservation of mass. Since $\rho = \text{constant}$, $\partial_t \rho = 0$, so the continuity equation is $\underline{v} \cdot (\rho \underline{u}) = 0$, i.e. the flux in must equal the flux out. Thus $u_1 A_1 = u_2 A_2$ or $u_2 = u_1 (A_1/A_2)$.

Now let $A_1, A_2, A_2 - A_1$ and W denote the areas shown



The momentum equation is $\partial_t (\rho \underline{u}) + \underline{v} \cdot (\rho \underline{u} \underline{u} - \underline{T}) = \underline{f}$

Now $\partial_t(\rho \underline{u}) = \rho \partial_t \underline{u}$. Immediately downstream of the enlargement, there will be turbulence so $\partial_t \underline{u} \neq 0$. The fluctuations will be about a steady average value $\langle \underline{u} \rangle$. The $\langle \cdot \rangle$ denotes a long-time average. Since $\partial_t \langle \underline{u} \rangle = 0$, and since there is no body force in the $\hat{x}_2$ direction, the time-averaged $\hat{x}_2$ component of the momentum equation is

$$\langle \underline{\nabla} \cdot (\rho \underline{u}_2 \underline{u} - T_{i2} \hat{x}_i) \rangle = 0.$$

Integrate this over the volume shown. By Gauss' theorem, letting $A_1 + A_2 + (A_2 - A_1) + W \equiv A$ we find

$$\left\langle \int_A n_i T_{i2} dA \right\rangle = \left\langle \int_A \rho u_2 \hat{n} \cdot \underline{u} dA \right\rangle.$$

Now $\hat{n} \cdot \underline{u} = 0$ on W and $A_2 - A_1$, so the

$$\text{r.h.s.} = \rho A_2 \overline{u_2^2} - \rho A_1 \overline{u_1^2} = \rho A_2 \overline{u_2} (\overline{u_2} - \overline{u_1}).$$

If we neglect viscous drag on W , the

$$\text{l.h.s.} = \int_{A_1 + A_2 + (A_2 - A_1)} -p n_2 dA = p_1 A_1 - p_2 A_2 + p_1 (A_2 - A_1)$$

$$= (p_1 - p_2) A_2.$$

Note we have set $p = p_1$ on $A_2 - A_1$. By continuity, that cannot be too far wrong. Note also that we can now remove the $\langle \cdot \rangle$, since nothing remaining depends on time. We get, equating,

$$(p_1 - p_2) A_2 = \rho A_2 \overline{u_2} (\overline{u_2} - \overline{u_1}) \text{ or}$$

$$p_2 = p_1 + \rho \overline{u_2} (\overline{u_1} - \overline{u_2})$$

3.(a). Let S be top surface of water (CD in picture), T surface of contact between tank (including outlet pipe) and water, and a cross-section across jet at AB in picture. Then $S+T+a$ is a closed surface containing only water. Traction $\hat{n} \cdot \underline{T} = \underline{0}$ on S (no atmospheric pressure while $\underline{T}$ vanishes, by assumption, on a). Hence

$$\int_{S+T+a} \hat{n} \cdot \underline{T} dA = \int_T \hat{n} \cdot \underline{T} dA, \text{ but minus this,}$$

$$\text{i.e. } - \int_T \hat{n} \cdot \underline{T} dA \equiv \underline{F} \text{ is the force}$$

exerted by the water on the tank.

Moreover,

$$\begin{aligned} \underline{F} &= - \int_{S+T+a} \hat{n} \cdot \underline{T} dA = - \int_V \underline{r} \cdot \underline{T} \cancel{dV} dV \\ &= \int_V [\underline{f} - \partial_t (\rho \underline{u}) - \underline{r} \cdot (\rho \underline{u} \underline{u})] dV. \end{aligned}$$

Now the only body force is gravity, so $\underline{f}_2 = 0$. Also, if $a \ll S$, the flow will be everywhere (inside the tank) steady to a very good approximation, so that $\partial_t (\rho \underline{u}) = \rho \partial_t \underline{u} = 0$. Thus

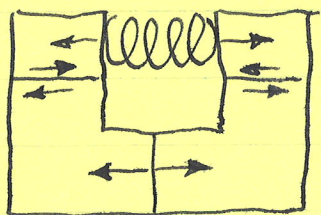
$$\underline{F}_2 = - \int_V \underline{r} \cdot (\rho u_2 \underline{u}) dV = - \int_{S+T+a} \rho u_2 \hat{n} \cdot \underline{u} dA.$$

Now $u_2 = 0$ on S , and $\hat{n} \cdot \underline{u} = 0$ on T , so we get only $\underline{F}_2 = - \int_a \rho u_2 \hat{n} \cdot \underline{u} dA = -\rho u^2 a$.

(b). Momentum flux (j^{th} component) is $\rho u_j \underline{u} - T_{ij} \hat{x}_i$. The leakage in the jet is $\rho u_j \underline{u} = \rho u_j u \hat{x}_2 = \rho u^2 \hat{x}_2 \delta_{j2}$. Only $\hat{x}_2$ component leaks out through the jet, and the amount is $\rho u^2 a$ units of momentum (gm cm s^{-1}) per second.

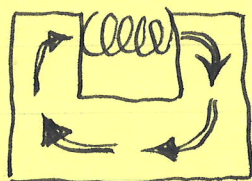
(c). The water in the tank loses $\hat{x}_2$ momentum at the rate $\rho u^2 a \text{ gm cm s}^{-1}$ per second through the jet, but gains momentum at the same rate because of the force exerted on it by the tank.

4. Suppose we draw force exerted by matter on + side of a surface on matter on - side as an arrow on the - side. We shall draw only $\hat{x}_2$ component of average traction at various places in the iron. The arrows all have ~~the~~ a length ~~equal~~ equal to the force $\underline{F}$ exerted by the spring $\div$ local cross-section of iron, since total force on any separate piece must vanish. The flux of $\hat{x}_2$ component of momentum is $-T_{i2} \hat{x}_i$.



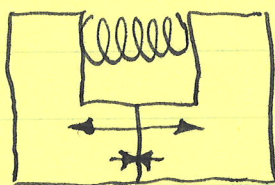
This component of momentum is flowing out of the right end of the spring and into the left end.

A very rough sketch of the momentum flow lines thus looks like. The $-\hat{x}_2$

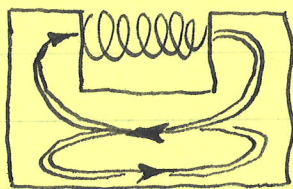


component of momentum is flowing in the opposite direction. Note that $\mathbf{r} \cdot \underline{\underline{\mathbf{T}}} = \underline{\underline{0}}$ in this

case. There are no sources or sinks of momentum in the iron. The exact flow lines can of course only be found by actually solving for T_{i2} in the block. Actually, just a little more thought shows that the rough picture above is not quite right. Consider the situation at the middle of the base of the block. It is clear that although the average stress is tensile, the stress near the bottom will be compressive, i.e.



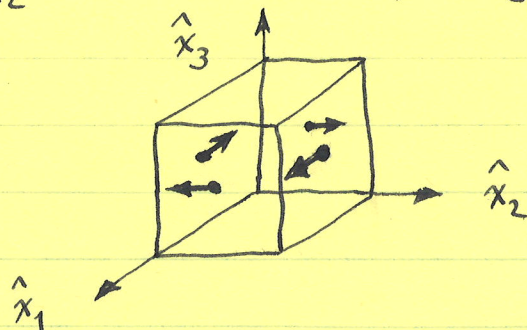
at the bottom is thus in the opposite direction, i.e.



5. Polyhedron, N faces, areas A_i , outward normals $\hat{n}_i$. Consider $\sum_i A_i \hat{n}_i = \int_{\partial V} \hat{n} dA$
- identity tensor
- $$= \int_{\partial V} \hat{n} \cdot \underline{\underline{\mathbf{I}}} dA = \int_V \mathbf{r} \cdot \underline{\underline{\mathbf{T}}} dV = \int_V \underline{\underline{0}} dV = \underline{\underline{0}}.$$

Energetic answers # 3.

1. $\underline{m} + \Lambda \underline{T} = \underline{0}$ or $m_i + \varepsilon_{ijk} T_{jk} = 0$.
 Thus $\varepsilon_{ilm} m_i + (\delta_{lj} \delta_{mk} - \delta_{lk} \delta_{mj}) T_{jk} = 0$
 or $\varepsilon_{ilm} m_i + T_{lm} - T_{ml} = 0$.
 Or, rewritten,
 $T_{jk} - T_{kj} = m_i \varepsilon_{ikj}$.
 Now $m_i = 10^6 \delta_{i3}$, so $T_{jk} - T_{kj} = 10^6 \varepsilon_{3kj}$.
 Thus $T_{13} - T_{31} = 0$ and $T_{23} - T_{32} = 0$ but
 $T_{12} - T_{21} = -10^6 \text{ dyne cm}^{-2} \approx -1 \text{ atm}$.
 If, in addition $T_{ij} + T_{ji} = 0$, then
 $T_{12} \approx -1/2 \text{ atm}$, $T_{21} \approx 1/2 \text{ atm}$.



2. $\Phi \equiv \int_{V(t)} \rho \Psi dV$.

Consider $\frac{d\Phi}{dt} = \frac{d}{dt} \int_{V(t)} \rho \Psi dV = \int_{V(t)} \rho \frac{D}{Dt} \Psi dV$

$= \int_{V(t)} \rho \underline{u} \cdot \underline{\nabla} \Psi dV = - \int_{V(t)} \underline{u} \cdot \underline{f} dV$
 $\uparrow$ since $\frac{\partial}{\partial t} \Psi = 0$.

The equation of conservation of energy may thus

be written $\frac{d}{dt} (K + \Pi + \Phi) = W_s + P - H$

includes now only
mechanical work due to
surface stresses.

The equation thus becomes

$$\partial_t [\rho(E + \Psi)] + \nabla \cdot [\rho(E + \Psi)\underline{u} + \underline{H} - \underline{u} \cdot \underline{T}]$$

$$= h$$

flux density $\nwarrow$ production rate

The point is that if $\underline{f} = -\nabla\Phi$ with $\partial_t\Phi = 0$ (e.g. the oceans or the atmosphere in the Earth's gravitational field), the concept "work done by body forces" can be replaced by the concept "mechanical potential energy".

3. We have $\partial_t(\rho E) + \nabla \cdot [\rho E \underline{u} + \underline{H} - \underline{u} \cdot \underline{T}] = h + \underline{u} \cdot \underline{f}$
and $\partial_t(\rho u) + \nabla \cdot [\rho u \underline{u} + \underline{H}] = \tau(\underline{T} \cdot \underline{\varepsilon}) + h$.
Subtracting and setting $K = \frac{1}{2} \rho u^2$,

$$\partial_t(\rho K) + \underbrace{\nabla \cdot [\rho K \underline{u} - \underline{u} \cdot \underline{T}]}_{\text{flux}} = \underbrace{\underline{u} \cdot \underline{f} - \tau(\underline{T} \cdot \underline{\varepsilon})}_{\text{production rate}}$$

Can alternatively write this in form

$$\frac{\partial}{\partial t}(\rho K) + \underbrace{\nabla \cdot (\rho K \underline{u})}_{\text{flux: all advective}} = \underbrace{\underline{u} \cdot (\underline{f} + \nabla \cdot \underline{T})}_{\text{production rate}}$$

Either interpretation is valid.

4. (a) If $\underline{T} = -p \underline{\underline{I}}$, then $\underline{T} : \underline{\underline{\varepsilon}} = -p \nabla \cdot \underline{u}$, so the internal energy equation becomes $p \nabla \cdot \underline{u} = 0$, but if p is constant, $D_t p = 0$ so $p \nabla \cdot \underline{u} = 0$ by continuity, so $\nabla \cdot \underline{u} = 0$.

(b) Current density of total energy is, since $\underline{H} = \underline{0}$, just $\underline{u} \left(\frac{1}{2} \rho u^2 + \rho \mathcal{U} \right) - \underline{u} \cdot \underline{T}$. In the jet $\underline{T}$ vanishes and $\underline{u} = V \hat{x}_2$. Thus the flux is $\hat{x}_2 \left(\frac{1}{2} \rho V^2 + \rho \mathcal{U} \right) V$. The rate of discharge of total energy is thus $aV \left(\frac{1}{2} \rho V^2 + \rho \mathcal{U} \right)$ ergs/sec.

(c) The total energy of the water in the tank decreases only due to the loss of the internal energy of the water. The loss of kinetic energy is negligible if $a \ll S$. To see this, note that

$$dK/dt = \frac{d}{dt} \int_{V(t)} \frac{1}{2} \rho u^2 dV$$

$$= \int_{\partial V(t)} \left(\frac{1}{2} \rho u^2 \right) \underline{q} \, dA, \quad \text{since } p = \text{constant} \quad \text{and } \frac{\partial}{\partial t} \underline{u} = 0 \quad (\text{the flow is assumed to be steady.})$$

here $\underline{q} \equiv$ outward normal velocity of $\partial V(t)$.

If $V(t)$ in the above equation is the volume enclosed by the surface $S + T + a$ (see answers #2), then $\partial V(t)$ consists only of the top surface S , on which $\underline{q} = \hat{n} \cdot \underline{u}$. But $\underline{u}$ on S is very small, of order $V(a/S)$, so dE/dt is of order $\frac{1}{2} \rho V^3 (a/S)^3 S = \frac{1}{2} \rho V^3 a (a/S)^2$. Since $a/S \ll 1$, this is $\ll$ than the first term in the rate of discharge of total energy. The rate at which the total energy within the tank changes is thus just $aV\rho U$ ergs/sec.

(d) The difference is $\frac{1}{2} \rho a V^3$. This much more energy squirts out through the jet than is lost by the tank per second. This must $\therefore$ be the rate at which body forces (gravity) do work on the water in the tank.

(e) + (f): Let z be measured upwards from the jet at $z=0$. The energy escaping through the jet is then still $(\frac{1}{2} \rho V^2 + \rho U) Va$ erg/sec, since the gravitational potential energy per unit volume, $\rho g z$, is zero at $z=0$. In time δt , the grav. pot. energy of the H_2O in the tank decreases by $\rho g H S (v \delta t)$, where $z=H$ at surface of water, ~~the~~ S = area of

top surface and v = downward velocity at top. But $v \delta t S$ = volume of H_2O lost in δt , and this is $V \delta t a$. The net decrease in ~~grav. pot. energy~~ grav. pot. energy in δt is thus $\rho g H a V \delta t$, i.e. the rate of decrease is $\rho g H a V$.

The net loss of energy thru the pipe must equal rate of decrease of total energy (k.e. + internal + grav. pot. energy) in the tank, so

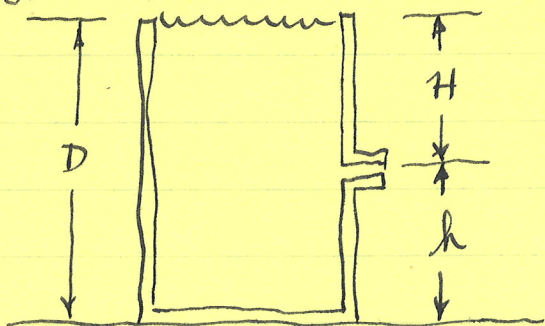
$$aV \left(\frac{1}{2} \rho V^2 + \rho U \right) = aV \rho U + aV \rho g H.$$

Hence

$$V = \sqrt{2gH}$$

The outlet velocity can thus be calculated knowing H , without discussing the details of the motion. This was first deduced by Torricelli, a pupil of Galileo. He is better known for his invention of the mercury barometer.

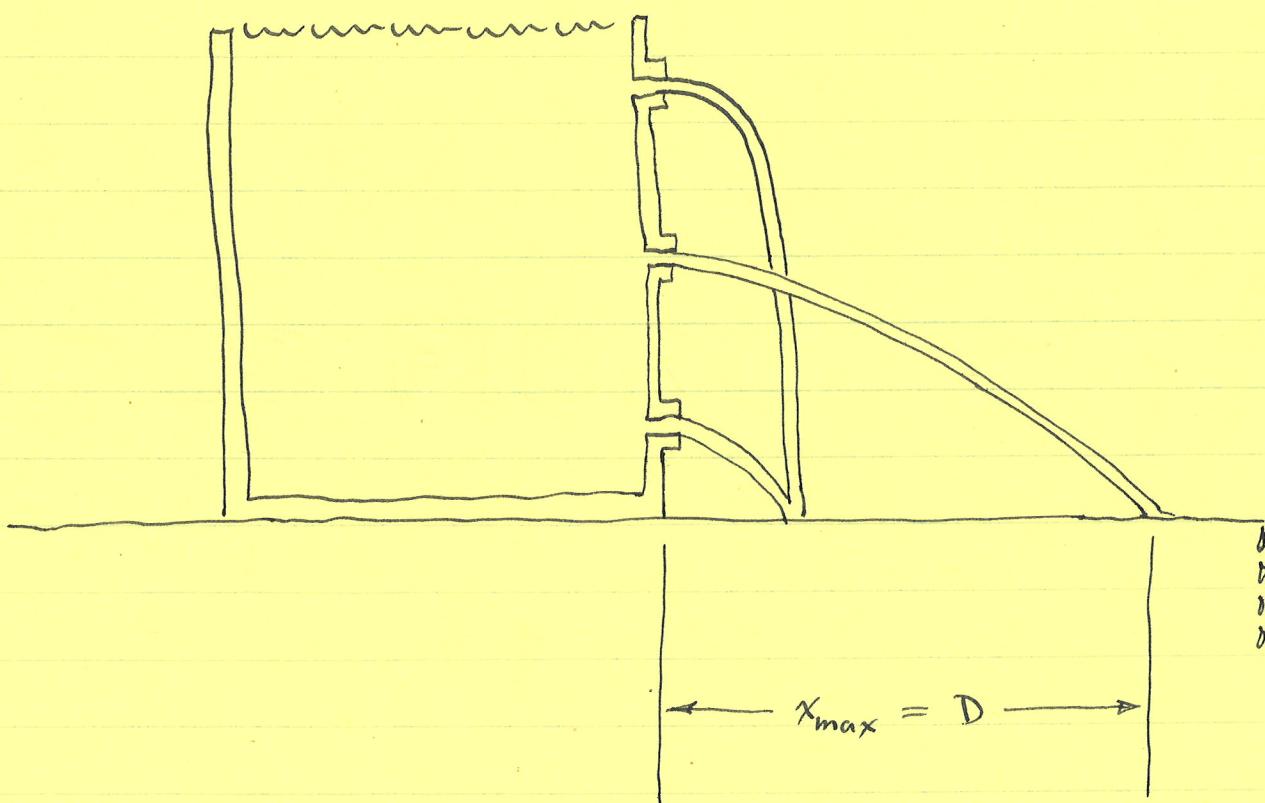
(g) Define H and h as shown below. The



water travels a vertical distance h to reach the table top. The time required, in free fall, is given by

$$h = \frac{1}{2} g t^2 \quad \text{or} \quad t = \sqrt{2h/g}.$$

Meanwhile, it is moving laterally with constant velocity V , so it travels a lateral distance $x = Vt = \sqrt{2gH} \cdot \sqrt{2h/g}$, or $x = 2\sqrt{Hh}$. Let $D = h + H$. Then $x = 2\sqrt{h(D-h)}$. To find $x_{\max}$, set $dx/dh = 0$. This gives $h = D/2$, a jet placed halfway up will shoot the farthest; in fact $x_{\max} = D$. For the situation shown, the trajectories thus look like:



The top and bottom jets hit at the same place since $\sqrt{Hh}$ is the same for both.

Ideal answers #4:

1. (a) For brevity let $CAS \equiv \{ \underline{0}, \hat{x}_1, \hat{x}_2, \hat{x}_3 \}$ and $CAS' \equiv \{ \underline{0}', \hat{x}_1', \hat{x}_2', \hat{x}_3' \}$. Let $\underline{r}$ be the position of a particle in CAS . In CAS' , it is $\underline{r}' = \underline{r} - \underline{R}$. Observers in the two frames will clearly agree on the volume of any region, the area of any surface, the force exerted on any body, and the mass of any body. Since the internal energy is the energy as observed in a reference frame moving everywhere with the local material velocity, they will agree on it as well. For a similar reason, they will agree on the heat flow vector and the heat production. Hence we have $\rho' = \rho$, $\underline{f}' = \underline{f}$, $\underline{T}' = \underline{T}$, $\underline{u}' = \underline{u}$, $\underline{H}' = \underline{H}$ and $\underline{h}' = \underline{h}$. Now consider relations between time derivatives. The substantial derivative $D_t \rho$ is the rate of change of ρ at a particular particle. Since $\rho' = \rho$, the substantial derivative is independent of reference frame, i.e. $D_{t'}' \rho' = D_t \rho$; likewise $D_{t'}' \underline{u}' = D_t \underline{u}$. Note: it is not true that $\partial_{t'}' \rho' = \partial_t \rho$; in fact $\partial_{t'}' \rho' = \partial_t \rho - \dot{\underline{R}}(t) \cdot \underline{r} \rho$, where $\dot{\underline{R}} = d\underline{R}/dt$. For $\underline{u}$, since $\underline{u}' \neq \underline{u}$, we have $D_{t'}' \underline{u}' = D_t \underline{u} - \underline{A}$, where $\underline{A} = \ddot{\underline{R}} = d^2 \underline{R}/dt^2$.

Now consider relations between spatial derivatives. From the chain rule, $\partial \rho / \partial r_i = \partial \rho' / \partial r_i = (\partial \rho' / \partial r_j') (\partial r_j' / \partial r_i) + (\partial \rho' / \partial t') (\partial t' / \partial r_i) = (\partial \rho' / \partial r_j') \delta_{ij}$ (since axes are in same direction) $= \partial \rho' / \partial r_i'$. Thus $\underline{\nabla} \rho = \underline{\nabla}' \rho'$; similarly $\underline{\nabla}' \cdot \underline{\underline{T}}' = \underline{\nabla} \cdot \underline{\underline{T}}$ and $\underline{\nabla}' \cdot \underline{\underline{H}} = \underline{\nabla} \cdot \underline{\underline{H}}$. We also have $\underline{\nabla}' \cdot \underline{\underline{u}}' = \underline{\nabla} \cdot \underline{\underline{u}}$, since $\partial u_i / \partial r_j = \partial u_i' / \partial r_j + \partial \dot{R}_i / \partial r_j$, but $\partial \dot{R}_i / \partial r_j = 0$, so the chain rule can be applied as before. Making use of all the above relations, we get

$$D_t' \rho' + \rho' \underline{\nabla}' \cdot \underline{\underline{u}}' = 0$$

$$\rho' D_t' \underline{\underline{u}}' = \underline{\nabla}' \cdot \underline{\underline{T}}' + (\underline{\underline{f}}' - \rho' \underline{\underline{A}})$$

$$\rho' D_t' \underline{\underline{u}}' + \underline{\nabla}' \cdot \underline{\underline{H}}' = \text{tr}(\underline{\underline{T}}' \cdot \underline{\underline{\varepsilon}}') + h'$$

(b). Once again we have $\rho' = \rho$, $\underline{\underline{u}}' = \underline{\underline{u}}$, $\underline{\underline{f}}' = \underline{\underline{f}}$, $D_t' \underline{\underline{u}}' = D_t \underline{\underline{u}}$, $D_t' \rho' = D_t \rho$, $h' = h$, $\underline{\underline{H}}' = \underline{\underline{H}}$, $\underline{\underline{T}}' = \underline{\underline{T}}$ and $\underline{\nabla}' \cdot \underline{\underline{T}}' = \underline{\nabla} \cdot \underline{\underline{T}}$ (the latter is true if ~~the axes are in the same direction~~), we choose CAS' so that $\hat{x}_i' = \hat{x}_i$ at the instant of interest). For $\underline{\underline{u}}$, we have $\underline{\underline{u}} = \underline{\underline{u}}' + \underline{\underline{\Omega}} \times \underline{\underline{r}}' \neq \underline{\underline{u}}'$. Nevertheless $\underline{\nabla}' \cdot \underline{\underline{u}}' = \underline{\nabla} \cdot \underline{\underline{u}}$ and $\underline{\underline{T}}' \cdot \underline{\underline{\varepsilon}}' = \underline{\underline{T}} \cdot \underline{\underline{\varepsilon}}$. To see this, write $u_i = u_i' + \varepsilon_{ijk} \Omega_j r_k'$ and $r_j' = r_j$. Then $\partial u_i / \partial r_j = (\partial / \partial r_j') [u_i' + \varepsilon_{ilk} \Omega_l r_k'] = \partial u_i' / \partial r_j' + \varepsilon_{ilk} \Omega_l \delta_{jk} = \partial u_i' / \partial r_j' + \varepsilon_{ilj} \Omega_l$. Thus $\underline{\nabla} \cdot \underline{\underline{u}} = \partial u_i / \partial r_i = \partial u_i' / \partial r_i' + \varepsilon_{ili} \Omega_l$, but $\varepsilon_{ili} = 0$, so $\underline{\nabla} \cdot \underline{\underline{u}} = \underline{\nabla}' \cdot \underline{\underline{u}}'$. Also $\underline{\underline{T}} \cdot \underline{\underline{\varepsilon}} = T_{ij} (\partial u_i / \partial r_j) = T_{ij}' (\partial u_i' / \partial r_j') + \Omega_l \varepsilon_{ilj} T_{ij}'$,

but $\varepsilon_{ijk} T_{ij}' = 0$, since $T_{ij}' = T_{ji}'$, so
 $\underline{T} \cdot \underline{\varepsilon} = \underline{T}' \cdot \underline{\varepsilon}'$. Putting all these results
 together, we get

$$\begin{aligned} \mathcal{D}_{t'}' \rho' + \rho' \underline{v} \cdot \underline{u}' &= 0 \\ \rho' [\mathcal{D}_{t'}' \underline{u}' + 2\underline{\Omega} \times \underline{u}' + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}') + \frac{d'\underline{\Omega}'}{d't'} \times \underline{r}'] \\ &= \underline{v}' \cdot \underline{T}' + \underline{f}' \end{aligned}$$

$$\rho' \mathcal{D}_{t'}' \underline{u}' + \underline{v}' \cdot \underline{H}' = \underline{T}' \cdot \underline{\varepsilon}' + h'.$$

The term $2\underline{\Omega} \times \underline{u}'$ is called the Coriolis acceleration, the term $\underline{\Omega} \times (\underline{\Omega} \times \underline{r}')$ is called the centripetal acceleration (after the famous Roman playwright Centripeto), and the term $\dot{\underline{\Omega}} \times \underline{r}'$ is called the Euler acceleration.

2. EUREKA! Case 1. Fluid of constant ρ .

$$\text{Net force} = \int_{\partial V} \hat{n} \cdot \underline{T} dA = \int_V \underline{v} \cdot \underline{T} dV$$

$$= - \int_V \underline{v} p dV = \int_V \rho \underline{v} \phi dV, \quad \text{equal and}$$

opposite to gravitational force acting on displaced fluid.

$$\text{Net torque} = \int_{\partial V} \underline{r} \times (\hat{n} \cdot \underline{T}) dA$$

$$= - \int_{\partial V} \hat{n} \cdot (\underline{T} \times \underline{r}) dA = - \int_V \underline{v} \cdot (\underline{T} \times \underline{r}) dV$$

$$= - \int_V (\underline{v} \cdot \underline{T}) \times \underline{r} dV + \int_V \cancel{\underline{v} \cdot \underline{T}}^{\text{zero}} dV = \int_V \underline{v} p \times \underline{r} dV$$

$$= \int_V \underline{r} \times \underline{\rho} \underline{r} \phi \, dV, \quad \text{equal and opposite to torque}$$

exerted on displaced fluid by gravitational field.

Case 2. Fluid density not necessarily constant. The only problem is that one must now decide how to specify the "density of the displaced fluid". But this is easy.

Surfaces of constant ϕ are well-defined throughout V , and outside of V , surfaces of constant ρ and ϕ coincide (note: the contribution of the rigid body itself to ϕ is ignored). Inside V , define the "density of displaced fluid" at any point $\underline{r}$ to be the density ρ associated with the surface $\phi = \text{const.}$ passing through $\underline{r}$. Then the above arguments all apply once again.

3. For an ideal polytropic gas, the equation of state is $p\tau = \tilde{R}\theta$. We also have $U = c_v \theta$ where $c_v = \text{specific heat at constant volume} = \text{const.}$

(a) $\theta = \theta_0$ (isothermal gas). $\rho(p) = p / \tilde{R}\theta_0$,

so

$$Q = \int_{p_0}^p \frac{dp}{\rho(p)} = \tilde{R}\theta_0 \int_{p_0}^p \frac{dp}{p} = \tilde{R}\theta_0 \ln(p/p_0).$$

(b) $S = S_0$ (isentropic gas). For an ideal polytropic gas $p\tau^\gamma = \text{const.} \exp(S/c_v)$.

Since $S = S_0$, $p\tau^\gamma = \text{const}$ or $\rho(p) = \text{const} \cdot p^{1/\gamma}$.
 In the reference state $p_0\tau_0 = \tilde{R}\theta_0$ or $\rho_0 = p_0/\tilde{R}\theta_0$,
 so $\text{const} = (1/\tilde{R}\theta_0) p_0^{\frac{\gamma-1}{\gamma}}$. Thus

$$\rho(p) = \left(\frac{p_0^{\frac{\gamma-1}{\gamma}}}{\tilde{R}\theta_0} \right) p^{1/\gamma}, \quad \text{so}$$

$$q = \int_0^p \frac{p \, dp}{\rho(p)} = \frac{\tilde{R}\theta_0}{p_0^{\frac{\gamma-1}{\gamma}}} \int_0^p p^{-1/\gamma} \, dp = \left(\frac{\gamma}{\gamma-1} \right) \tilde{R}\theta_0 \left(\frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}}.$$

↑ may be arbitrarily chosen

Note: in both (a) and (b), q is only defined (like all potential energies), to within an arbitrary constant.

4. (a) $\theta = \theta_0$ (isothermal atmosphere — a model first considered by Laplace).

$$q + \phi = \text{const} \quad \text{or} \quad \tilde{R}\theta_0 \ln(p/p_0) + g_0 z = \text{const}. \quad \text{Now } p = p_0 \text{ at } z = 0 \text{ (the ground)}$$

$$\Rightarrow \text{const} = 0. \quad \text{Thus}$$

$$p/p_0 = \rho/\rho_0 = \exp(-z/H) \quad \text{where } H = \text{scale height} = \tilde{R}\theta_0/g_0.$$

$\tilde{R} = R/\bar{M}$, $\bar{M}$ = ave. mol. wt. of air ≈ 29 , so
 $H = (8.3 \cdot 10^7/29)(273^\circ)(1/980) = 8 \cdot 10^5 \text{ cm}$
 or $H \sim 8 \text{ km}$ for air.

(b) $S = S_0$ (an isentropic atmosphere)

$$q + \phi = \text{const}, \quad \text{or}$$

$$\left(\frac{\gamma}{\gamma-1}\right) \tilde{R} \theta_0 \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} + g_0 z = \text{const.}$$

Since $p = p_0$ at $z = 0$, $\text{const} = \left(\frac{\gamma}{\gamma-1}\right) \tilde{R} \theta_0$, so

$$\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \left[\left(\frac{\gamma}{\gamma-1}\right) \tilde{R} \theta_0 - g_0 z \right] \left(\frac{\gamma-1}{\gamma}\right) \frac{1}{\tilde{R} \theta_0}, \text{ or}$$

$$\left(p/p_0\right)^{\frac{\gamma-1}{\gamma}} = 1 - \left(\frac{\gamma-1}{\gamma}\right) \frac{z}{H}, \text{ where } H = \text{the } \underline{\text{isothermal}} \text{ scale height.}$$

Also, since $p \tau^\gamma = \text{const}$, $p/p_0 = (\rho/\rho_0)^\gamma$, so

$$(\rho/\rho_0)^{\gamma-1} = 1 - \left(\frac{\gamma-1}{\gamma}\right) \frac{z}{H}.$$

This yields a model of the atmosphere having a finite thickness. Note that both p and ρ vanish at $z = z_{\text{crit}} = \frac{\gamma}{\gamma-1} H$. For air (a diatomic gas) $\gamma = c_p/c_v = \frac{7}{2} R / \frac{5}{2} R = 1.4$, so $z_{\text{crit}} = \frac{1.4}{0.4} (8) \approx 28 \text{ km.}$

5. (a) $\theta = \theta_0$ (isothermal atmosphere on a spherical planet). $g + \phi = \text{const}$, or $\tilde{R} \theta_0 \ln(p/p_0) - g_0 a^2/r = \text{const}$. Since $p = p_0$ at $r = a$, $\text{const} = -g_0 a$, so $p/p_0 = \rho/\rho_0 = \exp \left[\frac{g_0 a}{\tilde{R} \theta_0} \left(\frac{a}{r} - 1 \right) \right]$, or

$$p/p_0 = \rho/\rho_0 = \exp \left[\frac{a}{H} \left(\frac{a}{r} - 1 \right) \right].$$

Note as $r \rightarrow \infty$, $\frac{P}{P_0} = \frac{\rho}{\rho_0} \rightarrow e^{-a/H}$, a finite number, so the total mass of this atmosphere is infinite.

(b) $S = S_0$ everywhere $q + \phi = \text{const}$, or
 $\left(\frac{r}{a}\right)^{\frac{n}{n-1}} \tilde{R} \theta_0 \left(\frac{\rho}{\rho_0}\right)^{\frac{n-1}{n}} - g_0 a / r^2 = \text{const}.$

Evaluating at $r = a$, $\text{const} = \left(\frac{r}{a}\right)^{\frac{n}{n-1}} \tilde{R} \theta_0 - g_0 a$,

so
$$\left(\frac{P}{P_0}\right)^{\frac{n-1}{n}} = \left(\frac{\rho}{\rho_0}\right)^{n-1} = \frac{a}{H} \left(\frac{a}{r} - 1\right) \left(\frac{r-1}{r}\right) + 1.$$

In this case, both ϕ and ρ ^{vanish} at a finite r , namely at

$$r_{\text{crit}} = \frac{a}{1 - \left(\frac{r}{a}\right)^{\frac{n}{n-1}} \frac{H}{a}} \approx a \left[1 + \left(\frac{r}{a}\right)^{\frac{n}{n-1}} \frac{H}{a} \right],$$

or, for air on $\oplus$, $\left(\frac{r}{a}\right)^{\frac{n}{n-1}} H \approx 28 \text{ km}$ above $r = a$ (surface of $\oplus$).

$$\begin{aligned} \text{Total mass of atmosphere} &\equiv M = \int_a^{r_{\text{crit}}} 4\pi r^2 \rho(r) dr \\ &= \int_a^{r_{\text{crit}}} 4\pi r^2 \rho_0 \left[\frac{a}{H} \left(\frac{a}{r} - 1\right) \left(\frac{r-1}{r}\right) + 1 \right]^{1/n-1} dr. \end{aligned}$$

Ugh! Let's approximate this integral. Let $r - a = z$ where $z \ll a$. To first order in z/a , letting $z_{\text{crit}} = \frac{r}{n-1} H$,

$$M = 4\pi a^2 \rho_0 \int_0^{z_{\text{crit}}} \left[1 - \left(\frac{r-1}{r}\right) \frac{z}{H} \right]^{\frac{1}{n-1}} dz$$

$$= 4\pi a^2 \rho_0 \frac{\gamma H}{1-\gamma} \int_{z=0}^{z=z_{\text{ait}}} u^{\frac{1}{\gamma-1}} du, \quad \text{or}$$

$M = 4\pi a^2 H \rho_0$, the same as if the atmosphere were of constant density ρ_0 and height ~~the~~ H .

6. Sound propagation in a fluid. With the given assumptions, the equations governing the motion are

1. $\partial_t \rho + \underline{v} \cdot (\rho \underline{u}) = 0$, continuity
2. $\rho D_t \underline{u} + \underline{v} p = 0$, momentum
3. $D_t S = 0$, entropy.

It is convenient to obtain first an alternative form of eqn. 3, containing the same variables as in eqns. 1 and 2.

To do this, consider the equation of state $p = p(\tau, S)$. By differentiation, $dp = (\partial p / \partial \tau)_S d\tau + (\partial p / \partial S)_\tau dS \equiv -X d\tau + Y dS$ (this defines X and Y). Thus

$$\begin{aligned} D_t S &= \frac{1}{Y} (D_t p + X D_t \tau) \\ &= \frac{1}{Y} (D_t p - \frac{X}{\rho^2} D_t \rho). \end{aligned}$$

~~Now $X = -(\partial p / \partial \tau)_S = \rho^2 (\partial p / \partial \rho)_S$, so~~

Now $X = -(\partial p / \partial \tau)_S = \rho^2 (\partial p / \partial \rho)_S$, so

$D_t S = 0$ implies that

$$D_t p = (\partial p / \partial \rho)_S D_t \rho$$

Define a variable $c^2(p, \rho)$ by $c^2 \equiv (\partial p / \partial \rho)_S$.
Then eqn. 3 can be replaced by

$$3.^* \quad D_t p = c^2 D_t \rho$$

We shall show subsequently that c is the phase velocity of infinitesimal pressure disturbances (sound waves).

(a). In steady state $\underline{u} = \underline{0}$ so $D_t \underline{u} = \underline{0}$ and $\underline{\nabla} \cdot \underline{u} = 0$. Hence from eqn. 2, $\underline{\nabla} p = \underline{0}$. Thus pressure is constant throughout space, but may vary with time, i.e. $p = p_0(t)$.

But from eqn. 1, $D_t \rho = 0$ if $\underline{u} = \underline{0}$, and from eqn. 3*, $D_t p = 0$ if $D_t \rho = 0$.

But $D_t p = \partial_t p + \underline{u} \cdot \underline{\nabla} p = 0$, so $\partial_t p = 0$.

Thus p is independent of t also, i.e.

$p = \text{constant} \equiv p_0$. Note that $D_t \rho = 0$ implies that $\partial_t \rho = 0$ also in the initial state ($\underline{u} = \underline{0}$), but that $\underline{\nabla} \rho \neq 0$ in general, so the initial density ρ_0 can vary from place to place. The quantity c^2 will thus also vary from place to place.

(b). Now if the fluid is slightly disturbed:

$$1. \quad \partial_t (\rho_0 + \rho_1) + \underline{\nabla} \cdot [(\rho_0 + \rho_1) \underline{u}] = 0$$

$$2. \quad (\rho_0 + \rho_1) D_t \underline{u} = -\underline{\nabla} (p_0 + p_1)$$

$$3.^* \quad \partial_t (\rho_0 + \rho_1) + \underline{u} \cdot \underline{\nabla} (\rho_0 + \rho_1) = \frac{1}{c^2} [\partial_t (p_0 + p_1) + \underline{u} \cdot \underline{\nabla} (p_0 + p_1)].$$

Upon neglecting second order quantities, we find

$$1. \partial_t p_1 + \underline{v} \cdot (\rho \underline{u}) = 0$$

$$2. \rho_0 \partial_t \underline{u} + \underline{v} p_1 = 0$$

$$3^*. \partial_t p_1 + \underline{u} \cdot \underline{v} \rho_0 = \frac{1}{c^2} \partial_t p_1$$

Combine 1 and 3* to get

$$-\rho_0 \underline{v} \cdot \underline{u} = \frac{1}{c^2} \partial_t p_1. \quad \text{Now take } \partial_t \text{ to get } -\rho_0 \underline{v} \cdot \partial_t \underline{u} = (1/c^2) \partial_t^2 p_1, \text{ or using 2:}$$

$$\partial_t^2 p_1 = c^2 \rho_0 \underline{v} \cdot (\rho_0^{-1} \underline{v} p_1)$$

Here $c = c(\underline{r}) = \left(\partial p / \partial \rho \right)_s$ evaluated at p_0, ρ_0 (i.e. in the initial state). To evaluate c , let us define

$\kappa_s = -\frac{1}{\tau} \left(\frac{\partial \tau}{\partial p} \right)_s$, the isentropic compressibility, and

$\kappa_\theta = -\frac{1}{\tau} \left(\frac{\partial \tau}{\partial p} \right)_\theta$, the isothermal compressibility,

Also $\gamma \equiv c_p / c_v$. Now $\underline{X} = \rho^2 c^2 = \frac{\rho}{\kappa_s}$, so

$c^2 = \frac{1}{\rho \kappa_s}$ evaluated in initial state, i.e.

$$c^2 = \frac{1}{\rho_0 \kappa_s(p_0, \theta_0)}$$

Since $\kappa_s = \frac{\kappa_\theta}{\gamma}$, we may also write this in the form

$$c^2 = \frac{\gamma(p_0, \theta_0)}{\rho_0 \kappa_\theta(p_0, \theta_0)}$$

(c). Assume air is a diatomic ($\gamma = 1.4$) ideal gas, so that

$$p(p, S) = \text{const.} \cdot p^\gamma e^{S/cv}.$$

Then

$$\begin{aligned} (\partial p / \partial p)_S &= \gamma p^{\gamma-1} \cdot \text{const.} \cdot e^{S/cv} \\ &= \gamma p / p = \gamma p \tau. \end{aligned}$$

$$\text{Thus } c^2 = \gamma p_0 \tau_0 = \gamma \tilde{R} \theta_0$$

$$c^2 = \frac{\gamma R \theta_0}{M}$$

$$R/M = \frac{8.32 \cdot 10^7}{29}, \quad \theta_0 = 273^\circ \text{ K} \quad \text{gives}$$

$c \sim 330 \text{ m/sec}$, in good agreement with measured value.

Newton's famous error (corrected much later by Laplace) was to assume that sound propagation occurred isothermally, not

isentropically. He thus obtained

$$c^2(\text{Newton}) = \frac{1}{\rho_0 \kappa_0(p_0, \theta_0)} = \frac{R\theta_0}{M}.$$

His predicted value of c was thus a factor of $\sqrt{1.4}$ too low. The speed of sound had been measured in Newton's day, and he was aware of the discrepancy. He attributed it partly to "the crassitude of the solid particles of the air, by which the sound is propagated instantaneously" and partly to "the vapors in the air being of another spring". Oh, well.

(d). If $\rho_0 = \text{const}$, then $S = S_0 = \text{const}$; hence $c^2 = \text{const}$, so ~~the~~ the equation becomes

$$\boxed{\frac{\partial^2 p_1}{\partial t^2} = c^2 \nabla^2 p_1}$$

This is the wave equation. It is easy to verify that $p_1 = f(x-ct)$ and $p_1 = r^{-1} f(r-ct)$ are solutions. The factor of r^{-1} is the geometrical attenuation required to conserve energy.

Answers with a lot of potential #5.

1. Suddenly last summer, Cavitation.

(a). Potential flow at $t=0$; hence at all later times. Thus $\underline{u} = \nabla\psi$ and

$\nabla^2\psi = 0$ for $r > R(t)$. The b.c. is that

$\partial\psi/\partial r = \dot{R}(t)$ on $r = R(t)$. Try $\psi = A(t)/r$.

~~At~~ We know that $\nabla^2[A(t)/r] = 0$ (except at $r=0$). To satisfy b.c.

$A(t) = -R^2(t)\dot{R}(t)$. The solution to an exterior Neumann problem is unique, hence

$$\psi = -\frac{\dot{R}(t)R^2(t)}{r}$$

(b). Bernoulli's theorem $\frac{P}{\rho} + \partial_t\psi + \frac{1}{2}|\nabla\psi|^2 = f(t)$

At $r = \infty$, $p = P$ and at $r = R(t)$, $p = 0$. Thus $f(t) = P/\rho$, and

$$\begin{aligned} \frac{P}{\rho} &= \left[-\partial_t\psi - \frac{1}{2}|\nabla\psi|^2 \right]_{r=R(t)} \\ &= -\frac{\ddot{R}R^2}{R} - \frac{2\dot{R}^2R}{R} + \frac{1}{2}\left(\frac{\dot{R}R^2}{R^2}\right), \quad \text{or} \end{aligned}$$

$$\ddot{R}R + \frac{3}{2}(\dot{R})^2 + \frac{P}{\rho} = 0. \quad *$$

$$(c). \quad \frac{d}{dt}(R^3\dot{R}^2) = 2R^2\dot{R}(R\ddot{R} + \frac{3}{2}\dot{R}^2)$$

$$= -\frac{2P}{\rho}R^2\dot{R} = -\frac{d}{dt}\left(\frac{2P}{\rho}\frac{R^3}{3}\right).$$

Thus $R^3 \dot{R}^2 + \frac{2}{3} \frac{P}{\rho} R^3 = \text{constant}$.

Since $\dot{R}(0) = 0$ at $R(t) = a$, we get

$** \quad R^3 \dot{R}^2 = \frac{2}{3} \frac{P}{\rho} (a^3 - R^3)$. This is a first integral of $*$.

(d). Apply Bernoulli's theorem at an arbitrary radius r .

$$\frac{p(r) - P}{\rho} = \frac{\ddot{R} R^2}{r} + \frac{2 \dot{R}^2 R}{r} - \frac{1}{2} \left(\frac{\dot{R} R^2}{r^2} \right)^2$$

Using ~~the first integral~~ $*$ and $**$, this becomes

$$p(r) = P - \frac{1}{3} P \left[\frac{R}{r^4} (a^3 - R^3) + \frac{1}{R^2 r} (4R^3 - a^3) \right]$$

To find the maximum, set $dp/dr = 0$.

This gives

$$r^3 = 4R^3 \left[\frac{a^3 - R^3}{a^3 - 4R^3} \right]$$

Now $a^3 - R^3 \geq 0$ for all t .

Hence until $R(t) = a/4^{1/3} \sim a/1.6$, the max. pressure is at $r = \infty$. When $R(t) \leq \frac{a}{1.6}$ the max. pressure is at

$$r \approx 1.6 R \left[\frac{x^3 - 1}{x^3 - 4} \right]^{1/3} \quad \text{where } x \equiv \frac{a}{R}.$$

$$(e) \quad \dot{R} = \sqrt{\frac{2}{3} \frac{P}{\rho}} \left(\frac{a^3}{R^3} - 1 \right)^{1/2} \quad \text{Hence}$$

$$t = \sqrt{\frac{3}{2} \frac{\rho}{P}} \int_a^{R(t)} \left(\frac{a^3}{R^3} - 1 \right)^{-1/2} dR$$

To find time until complete collapse, set $R(t) = 0$.

Integrate by substitution (let $x = (R/a)^3$ and $dx = \frac{3R^2}{a^3} dR$). Then

$$t_{\text{coll}} = a \sqrt{\frac{\rho}{6P}} \int_0^1 x^{-1/6} (1-x)^{-1/2} dx.$$

This is the integral which defines the β function (see, e.g. Lamb p. 122)

$$t_{\text{coll}} = a \sqrt{\frac{\rho}{6P}} \frac{\Gamma(5/6) \Gamma(1/2)}{\Gamma(4/3)} \approx .915 a \sqrt{\frac{\rho}{P}}$$

$$(f). \quad V = P \frac{4}{3} \pi R^3 \quad \text{and} \quad T = \frac{1}{2} \rho \int_{r \geq R} |\nabla \psi|^2 dV$$

$$= \frac{1}{2} \rho R^4 \dot{R}^2 \int_R^\infty 4\pi r^2 \frac{dr}{r^4} = 2\pi \rho R^3 \dot{R}^2$$

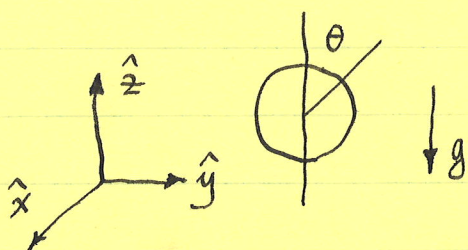
$$\text{The Lagrangian} \quad L = T - V = 2\pi R^3 \left[\rho \dot{R}^2 - \frac{2}{3} P \right]$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{R}} \right) - \frac{\partial L}{\partial R} = 0 \quad \text{gives} \quad \frac{3}{2} \dot{R}^2 + R^3 \ddot{R} + \frac{P}{\rho} = 0$$

which is $\star$. Its first integral, we have seen is $\star\star$. We now see that $\star\star$ is a statement of the conservation of energy, since $T + V = 2\pi \rho \left[R^3 \dot{R}^2 + \frac{2}{3} \frac{P}{\rho} R^3 \right] = \frac{4}{3} \pi P a^3$.

2. The ping-pong ball.

Let $s(t) \equiv$ upward displacement of ping-pong ball, with $s(0) = 0$.



The resulting water flow is irrotational by Helmholtz's theorem.

$$\underline{u} = \nabla \psi$$

$$\nabla^2 \psi = 0 \text{ in } r \geq a \text{ with}$$

$$\partial \psi / \partial r = \dot{s} \cos \theta \text{ on } r = a.$$

The solution to this exterior Neumann problem is $\psi = -\frac{1}{2} \dot{s} a^3 \cos \theta / r^2$.

(a). With Lagrangian.

$$\begin{aligned} T_{\text{fluid}} &= \frac{1}{2} \int_{E-V} \rho |\nabla \psi|^2 dV = -\frac{1}{2} \rho \int_{\partial V} \psi (\partial \psi / \partial r) dA \\ &= \frac{1}{3} \pi \rho a^3 \dot{s}^2. \end{aligned}$$

$$V_{\text{fluid}} = - \left(M_{\text{displaced fluid}} \right) g s = - \frac{4}{3} \pi a^3 \rho g s.$$

The ball itself has $T_{\text{ball}} = V_{\text{ball}} = 0$ since it is massless. Thus

$$L = T - V = T_{\text{fluid}} - V_{\text{fluid}} = \frac{1}{3} \pi \rho a^3 (\dot{s}^2 + 4gs)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{s}} \right) - \frac{\partial L}{\partial s} \text{ reduces to } \ddot{s} = 2g.$$

The ball accelerates upward with ~~the~~ twice

the acceleration of gravity!

Another way of saying the same thing: the ball moves as if it had an inertial mass = $\frac{1}{2}$ the mass of the displaced fluid.

(b). Now without a Lagrangian, use Bernoulli's theorem to find the pressure acting on the ball.

$$\frac{p}{\rho} = -\partial_t \psi - \frac{1}{2} |\nabla \psi|^2 - \phi + f(t)$$

Now at ∞ where $\psi = 0$, say that $\phi = \phi_\infty$ and that $\phi = 0$ (this reference level can either be chosen way above the ball or way below it). Then

$$\begin{aligned} \frac{p - p_\infty}{\rho} &= -\partial_t \psi - \frac{1}{2} |\nabla \psi|^2 - gz \\ &= -\frac{1}{2} a \cos \theta \ddot{s} + \underbrace{\frac{1}{2} \dot{s}^2 \left(\frac{9}{4} \cos^2 \theta - \frac{5}{4} \right)}_{\text{this term integrates to zero}} - gz \end{aligned}$$

this term integrates to zero
(D'Alembert's "paradox")

$$\text{Total force exerted by fluid on ball} = \int_{\partial V} p \hat{n} dA$$

The term $-gz$ integrates to the static buoyant force (Archimedes' principle) $\rho g V \hat{z}$. The first term integrates to $-\frac{1}{2} \rho V \ddot{s} \hat{z}$. But force = mass $\cdot$ acceleration = 0, so $-\frac{1}{2} \rho V \ddot{s} + \rho g V = 0$ or $\ddot{s} = 2g$.

3. (a). The outer surface is $r = a + \xi$ where

$$\xi = \sum_l \sum_m \xi_l^m Y_l^m.$$

Write $\phi = \phi_0 + \delta\phi$ where ϕ_0 is the gravitational potential in the absence of deformation, i.e.

$$\phi_0 = \begin{cases} -\frac{GM}{2a^3} (3a^2 - r^2) ; r \leq a \\ -\frac{GM}{r} ; r \geq a \end{cases}$$

The perturbation $\delta\phi$ must satisfy $\nabla^2 \delta\phi = 0$ everywhere, so

$$\delta\phi = \sum_l \sum_m A_l^m r^l Y_l^m ; r \leq a \quad \text{and}$$

$$\delta\phi = \sum_l \sum_m B_l^m r^{-l-1} Y_l^m ; r \geq a$$

The coefficients A_l^m and B_l^m can be determined from the jump conditions

at $r = a$: $[\delta\phi]_{\pm} = 0$ and

$[\partial_r \delta\phi]_{\pm} = 4\pi G \rho \xi$ (the "surface mass density" associated with the displacement ξ is $\rho \xi$)

This gives

$$A_l^m = \frac{3GM}{(2l+1)a^{l+2}} \xi_l^m$$

$$B_l^m = \frac{3GM}{(2l+1)} a^{l-1} \xi_l^m.$$

The value of ϕ on the deformed surface

is , to first order in ξ ,

$$\phi_{\text{surface}} = - \frac{GM}{a+\xi} + \delta\phi_{r=a}$$

$$\approx - \frac{GM}{a} \left[1 - \frac{\xi}{a} \right] + \delta\phi_{r=a} , \text{ or}$$

$$\phi_{\text{surface}} = - \frac{GM}{a} \left[1 - \sum_l \sum_m \frac{2(l-1)}{2l+1} \left(\frac{\xi_l^m}{a} \right) Y_l^m \right] ,$$

(b) . The fluid motion will be a potential flow ; $\underline{u} = \nabla\psi$ with $\nabla^2\psi = 0$ in $r=a$.

Hence

$$\psi = \sum_l \sum_m \left(\frac{r}{a} \right)^l \psi_l^m(t) Y_l^m .$$

We need to find the coefficients $\psi_l^m(t)$.
The linearized kinematical b.c. (applied on $r=a$) is $\partial_t \xi = \partial_r \psi$ or

$$\partial_t \xi_l^m = \frac{l}{a} \psi_l^m . \quad *$$

We also have Bernoulli's theorem, which since $p=0$ on the free outer surface , takes the form , to first order,

$$\partial_t \psi + \phi_{\text{surface}} = \text{constant} .$$

The term $\frac{1}{2} |\nabla\psi|^2$ can be neglected, as

it is of second order, we thus get

$$\partial_t^2 \psi_l^m + \frac{2l(l-1)}{2l+1} \left(\frac{g}{a} \right) \psi_l^m = \text{constant}, \quad **$$

where $g = \frac{4}{3} \pi G a$.

Combining * and **, we find

$$\partial_t^2 \psi_l^m + \frac{2l(l-1)}{2l+1} \frac{g}{a} \psi_l^m = 0 \quad \text{and}$$

$$\partial_t^2 \xi_l^m + \frac{2l(l-1)}{2l+1} \frac{g}{a} \xi_l^m = 0.$$

This is the equation of motion of a simple harmonic oscillator with squared eigenfrequency

$$\omega_l^2 = \frac{2l(l-1)}{2l+1} \frac{g}{a} = \frac{2l(l-1)}{2l+1} \left[\frac{4}{3} \pi G \rho \right]$$

These are the eigenfrequencies of a self-gravitating incompressible fluid sphere, first deduced by Kelvin.

There is no dependence of ω_l^2 on the order m , because of spherical symmetry. There is thus a $2l+1$ degeneracy.

(c). $l=0$ is impossible for an incompressible fluid, since it implies a net

change of volume.

$l=1$ is simply rigid body translation.

Both $l=0$ and $l=1$ have zero frequency, as expected.

(d). The fundamental is $l=2$ (denoted ${}_0S_2$). For earth-like values, $\rho = 5517 \text{ kg/m}^3$ and $a = 6371 \text{ km}$, find $T \sim 94$ minutes. The period of ${}_0S_2$ for the actual Earth is ~ 54 minutes. Elasticity, which has been neglected in the above calculation, acts as an additional restoring force and shortens the period.

4. Capillary waves.

(a). Suppose a surface is defined by an equation of the form $\Psi(\underline{r}) = 0$. The unit normal is then given by $\hat{n} = \underline{\nabla}\Psi / |\underline{\nabla}\Psi|$, or

$$n_k = \frac{\partial \Psi}{\partial r_k} \left(\sum_j \frac{\partial \Psi}{\partial r_j} \frac{\partial \Psi}{\partial r_j} \right)^{-1/2}.$$

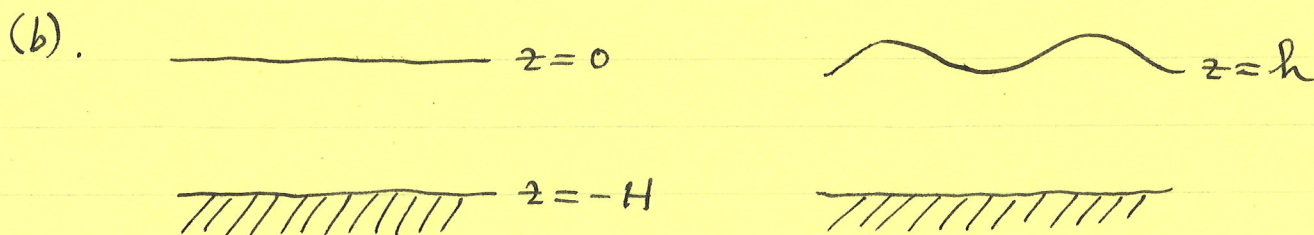
Then $\underline{\nabla} \cdot \hat{n} = \partial n_k / \partial r_k$ is

$$\underline{\nabla} \cdot \hat{n} = \partial n_k / \partial r_k = \frac{|\underline{\nabla}\Psi|^2 (\nabla^2 \Psi) - \sum_{ij} \frac{\partial \Psi}{\partial r_i} \frac{\partial \Psi}{\partial r_j} \frac{\partial^2 \Psi}{\partial r_i \partial r_j}}{|\underline{\nabla}\Psi|^3}.$$

In this case $\Psi = z - h(x, y)$ so $\underline{\nabla}\Psi = \hat{z} - \underline{\nabla}h$.
Then

$$\underline{v} \cdot \hat{n} = \frac{(1 + |\underline{v}h|^2)(-\nabla^2 h) + \sum_{ij} \frac{\partial h}{\partial r_i} \frac{\partial h}{\partial r_j} \frac{\partial^2 h}{\partial r_i \partial r_j}}{(1 + |\underline{v}h|^2)^{3/2}}.$$

Neglecting terms of second order in $\underline{v}h$, we get simply $\underline{v} \cdot \hat{n} = -\nabla^2 h = -\partial_x^2 h - \partial_y^2 h$.



We consider the linearized problem $\underline{u} = \underline{\nabla}\Psi$, $\nabla^2 \Psi = 0$ in $-H \leq z \leq 0$ and $\partial_z \Psi = 0$ on the bottom bdry $z = -H$.

What are the bdry conditions on $z = 0$.

The kinematical b.c. is unchanged:

$$\partial_t h = \partial_z \Psi.$$

The dynamical ~~kinematical~~ b.c., which in the absence of surface tension was

$$p = p_A \text{ is now } p = p_A + \gamma \underline{v} \cdot \hat{n}$$

$$\text{or } p = p_A - \gamma \nabla^2 h. \quad (*)$$

Bernoulli's theorem still applies in the usual form

$$\partial_t \Psi + \frac{1}{2} |\underline{\nabla}\Psi|^2 + gz + \frac{p}{\rho} = \frac{p_A}{\rho}, \text{ or when}$$

linearized and applied on $z = h$:

$$\partial_t \Psi + gh + \frac{p}{\rho} = \frac{p_A}{\rho}.$$

Combining this with (*) we get

$$\partial_t \psi + gh - \rho^{-1} \gamma \nabla^2 h = 0.$$

When we look for wavelike solutions

$h(x, y, t) = A \exp [i (lx + my - \omega t)]$, this becomes

$$\partial_t \psi + (g + \rho^{-1} \gamma k^2) h = 0 \quad \text{where} \\ k^2 = l^2 + m^2 \text{ is the horizontal wave number.}$$

It is now simplest to notice that the above is exactly the same as in the case without surface tension, with g replaced by $-g + \rho^{-1} \gamma k^2$. The effect of surface tension is thus exactly the same as the effect of such an increase in gravity. Short waves are obviously affected more. With the substitution $g \rightarrow \cancel{g} + \rho^{-1} \gamma k^2$, the dispersion relation is

$$\omega^2 = (gk + \rho^{-1} \gamma k^3) \tanh kH.$$

(c). In deep H_2O ($kH \gg 1$), this reduces to $\omega^2 = gk + \rho^{-1} \gamma k^3$. The phase speed c is

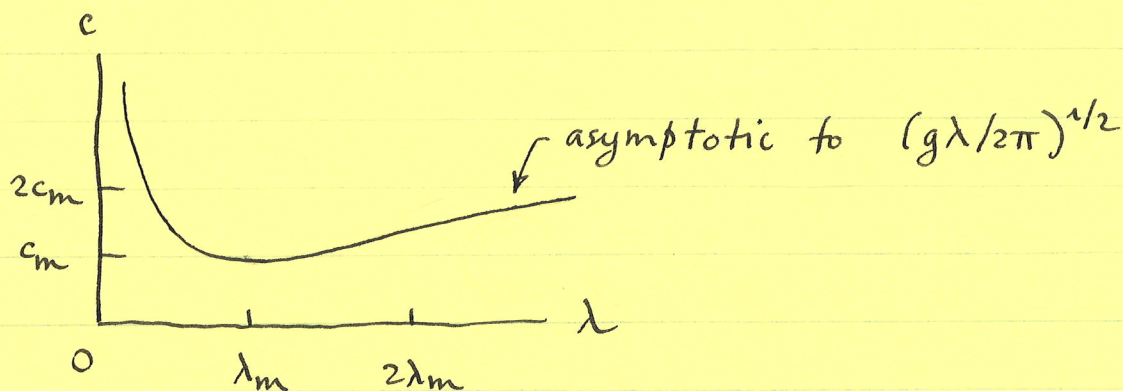
given by $c = \omega/k = [(g + \rho^{-1} \gamma k^2)/k]^{1/2}$.

This has a minimum at $k_{\min} = (\rho g / \gamma)^{1/2}$

or $\lambda_{\min} = 2\pi (\gamma / \rho g)^{1/2} = 1.7 \text{ cm}$ for water.

The corresponding $c_{\min}$ is $c_{\min} = (2g / k_{\min})^{1/2} = 23 \text{ cm/s}$ for water. A plot looks

like this: We can write c in terms of c_m and λ_m in the form $c^2 = \frac{1}{2} c_m^2 \left(\frac{\lambda}{\lambda_m} + \frac{\lambda_m}{\lambda} \right)$.



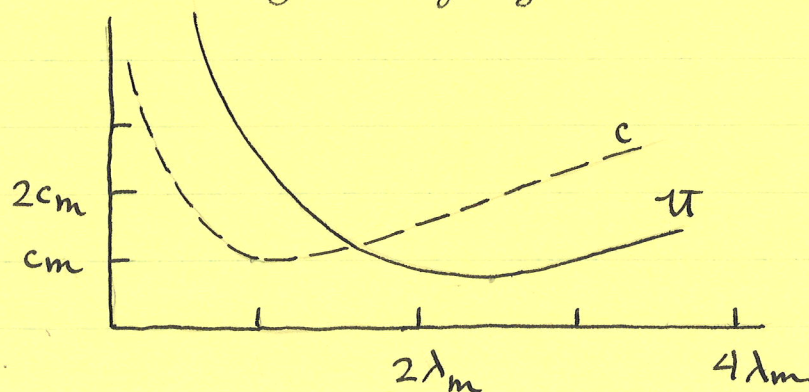
$$(d) \quad u = dw/dk = d(ck)/dk \\ = c + k dc/dk = c - \lambda dc/d\lambda.$$

Using this, u can be written most conveniently in the form

$$u = c \left[1 - \frac{1}{2} \frac{\lambda^2 - \lambda_m^2}{\lambda^2 + \lambda_m^2} \right].$$

Thus ~~when~~ when $\lambda \gg \lambda_m$, $u = \frac{1}{2}c$ as ~~for~~ for ordinary gravity waves. When $\lambda \ll \lambda_m$, $u = \frac{3}{2}c$.

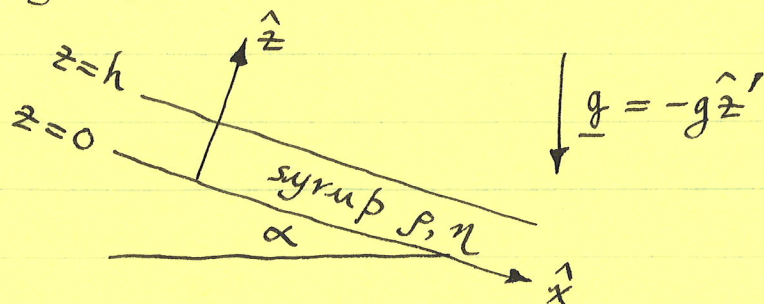
The minimum group velocity occurs at $\lambda = 2.54 \lambda_m$ and is given by $u = .767 c_m$. A plot thus looks like (very roughly). The c curve



is shown also (dotted).

Answers # 6

1. Pouring syrup on an inclined pancake.



- (a). Assume $\partial/\partial x = \partial/\partial y = 0$ and $\partial/\partial t = 0$.

Then $\underline{v} \cdot \underline{u} = 0 \Rightarrow w = \text{constant}$, but

$$w(0) = 0 \Rightarrow w(z) = 0 \Rightarrow \underline{u} \cdot \underline{\nabla u} = \underline{0},$$

so the NS equation becomes

$$\begin{aligned} & (\rho g \sin \alpha + \eta \partial^2 u / \partial z^2) \hat{x} \\ & - (\partial p / \partial z + \rho g \cos \alpha) \hat{z} = \underline{0}. \end{aligned}$$

The $\hat{z}$ component implies that

$$p(z) = -\rho g z \cos \alpha + \text{const}, \text{ but}$$

$$p(h) = p_A, \text{ so}$$

$$p(z) = p_A + \rho g (h - z) \cos \alpha.$$

The $\hat{x}$ component implies that

$$u(z) = (\rho g / 2\eta) \sin \alpha z^2 + Az + B.$$

The b.c. are that $u(0) = 0$ and

$$\partial u / \partial z (h) = 0, \text{ so}$$

$$u(z) = (\rho g \sin \alpha / 2\eta) z (2h - z).$$

- (b). $\underline{T} = -p \underline{\underline{I}} + 2\eta \underline{\underline{D}}.$

$$= -p \underline{\underline{I}} + \eta (\partial u / \partial z) (\hat{x} \hat{z} + \hat{z} \hat{x})$$

$$\underline{T} = - [p_A + \rho g \cos \alpha (h-z)] \underline{I} + [\rho g \sin \alpha (h-z)] [\hat{x}\hat{z} + \hat{z}\hat{x}].$$

(c). The traction exerted on the incline is

$$\hat{z} \cdot \underline{T}(0) = \hat{x} [\rho g h \sin \alpha] - \hat{z} [p_A + \rho g h \cos \alpha].$$

(d). The rate of heat generation is

$$\begin{aligned} \text{tr}(\underline{T} \cdot \underline{\varepsilon}) &= 2 T_{xz} \varepsilon_{xz} \\ &= (1/\eta) [\rho g (h-z) \sin \alpha]^2. \end{aligned}$$

(e). The internal energy equation is $\rho D_t \mathcal{U} + \nabla \cdot \underline{H} = \text{tr}(\underline{T} \cdot \underline{\varepsilon})$. Clearly the temperature θ will be a function only of z . Since there is no $\hat{z}$ component of the flow, it is clear also that $D_t \mathcal{U} = \partial_t \mathcal{U} + \underline{u} \cdot \nabla \mathcal{U} = 0$. Thus, with the stipulation $\underline{H} = -\kappa \nabla \theta$, we must solve

$$\partial^2 \theta / \partial z^2 = - (1/\kappa \eta) (\rho g \sin \alpha)^2 (z-h)^2.$$

Thus

$$\begin{aligned} \theta(z) &= - (1/12 \kappa \eta) (\rho g \sin \alpha)^2 (z-h)^4 \\ &\quad + Az + B. \end{aligned}$$

The b.c. are that $\theta(0) = \theta(h) = \theta_0 \equiv 20^\circ \text{C}$.

Thus $B = (1/12 \kappa \eta) (\rho g \sin \alpha)^2 h^4 + \theta_0$ and

$$A = - (1/12 \kappa \eta) (\rho g \sin \alpha)^2 h^3.$$

$$\text{Finally } \theta(z) = \theta_0 + (1/12 \kappa \eta) (\rho g \sin \alpha)^2 [h^4 - h^3 z - (z-h)^4].$$

2. Spin-up of a bucket. Use cylindrical coordinates r, θ, z . Let P denote the modified pressure, defined by $p = -\rho g z + P$. The laminar assumption is that $\partial/\partial z = \partial/\partial \theta = 0$. Then $\nabla \cdot \underline{u} = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r u_r) = 0 \Rightarrow r u_r = \text{constant}$, but $u_r = 0$ on $r = a$, so $u_r = 0$. From physical considerations, we shall also require that $u_z = 0$. Let us now use u to denote the only non-zero component of the flow, namely u_θ . The NS equation in cylindrical coordinates becomes

$$\hat{r}: (1/\rho) \partial P / \partial r = u^2 / r$$

$$\hat{\theta}: \partial u / \partial t = \nu \left(\partial^2 u / \partial r^2 + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right).$$

The first equation can be used to find P once u is known. The boundary and initial conditions are

$$u(a, t) = \Omega a \quad \text{for } t > 0$$

$$u(r, 0) = 0 \quad \text{for } 0 \leq r \leq a.$$

It is slightly more convenient to consider the function $w(r, t)$, defined by

$$w \equiv \Omega r - u.$$

The PDE satisfied by w is the same as that satisfied by u , viz.

$$\partial w / \partial t = \nu \left(\partial^2 w / \partial r^2 + r^{-1} \partial w / \partial r - r^{-2} w \right)$$

with b.c. $w(0, t) = w(a, t) = 0$.

The solution to this b.v. problem is the Fourier-Bessel series

$$w(r, t) = \sum_{n=1}^{\infty} A_n J_1(\lambda_n r/a) \exp(-\lambda_n^2 \nu t/a^2)$$

where λ_n are the zeroes of $J_1(\lambda)$, i.e.

$$J_1(\lambda_n) = 0, \quad n = 1, 2, \dots$$

The initial condition $w(r, 0) = \Omega r$ will be satisfied if

$$\sum_{n=1}^{\infty} A_n J_1(\lambda_n r/a) = \Omega r.$$

Using Bessel function orthogonality to solve for A_n , we find

$$A_n = - \frac{2\Omega a}{\lambda_n J_0(\lambda_n)}.$$

The velocity distribution is therefore

$$u(r, t) = \Omega r + 2\Omega a \sum_{n=1}^{\infty} \frac{J_1(\lambda_n r/a)}{\lambda_n J_0(\lambda_n)} \exp(-\lambda_n^2 \nu t/a^2).$$

The first term represents the final rigid body rotation.

(b) + (c). The departure from rigid body rotation, which decays exponentially, represents the growth inward of the boundary layer. The first ~~term~~

two roots of $J_1(\lambda)$ are $\lambda_1 \sim 3.83$ and $\lambda_2 \sim 7.02$. Since $J_1(\lambda_n r/a) / J_0(\lambda_n) = O(1)$, and since it is λ_n^2 which appears in the argument of the exponential, it is clear that the departure of $u(r, t)$ from Ωr is dominated by the first term in the sum. The water will be essentially co-rotating with the pail when (say) $\lambda_1^2 \nu t / a^2 = 2$, so that $\exp(-\lambda_1^2 \nu t / a^2) = e^{-2} \sim 0.135$.

(d). The water (or coffee) will be almost 90% of the way to its final state after $t = 2a^2 / \lambda_1^2 \nu$ seconds. For $a = 4$ cm, $\nu = 10^{-2}$ cm²/s, this is $t = 218$ s = 3.6 minutes.

(e). Casual observation discloses that the actual damping time in a coffee cup is about an order of magnitude less than the above prediction. The dominant spin-up mechanism is in fact controlled by Ekman layer suction at the cup's bottom. For a good discussion of this rather complicated process, see Greenspan, The Theory of Rotating Fluids, Cambridge University Press, 1969.

3. Sound attenuation.

(a). The velocity field is $\underline{u} = u_0 \hat{x} \cos(kx - \omega t)$.

For the density, pressure and temperature, we write ~~$\rho = \rho_0 + \rho_1$~~ $\rho = \rho_0 + \rho_1$,

$$p = p_0 + p_1, \quad \theta = \theta_0 + \theta_1.$$

From problem 6 of problem set #5, we have the linearized relations

$$1. \quad \partial_t \rho_1 + \rho_0 \nabla \cdot \underline{u} = 0$$

$$2. \quad \partial_t p_1 = c_0^2 \partial_t \rho_1.$$

We have in addition used the fact that $\nabla p_0 = 0$ to simplify the above two relations. Equation 1 implies that $\rho_1 = (\rho_0 k / \omega) u_0 \cos(kx - \omega t)$, or since $c_0 = \omega / k$,

$$\rho_1 = (\rho_0 / c_0) u_0 \cos(kx - \omega t).$$

Equation 2 then implies that

$$p_1 = (\rho_0 c_0) u_0 \cos(kx - \omega t).$$

Since the sound wave is propagating isentropically, θ_1 is given by

$$\theta_1 = (\partial \theta / \partial p)_s p_1.$$

But by one of Maxwell's relations for a fluid:

$$\begin{aligned} (\partial \theta / \partial p)_s &= - (\partial s / \partial p)_\theta / (\partial s / \partial \theta)_p \\ &= (\partial \tau / \partial \theta)_p / (\partial s / \partial \theta)_p \end{aligned}$$

$$= (\theta / c_p) (\partial \tau / \partial \theta)_p$$

$$= \alpha \tau \theta / c_p \quad \text{where}$$

$c_p = \theta (\partial s / \partial \theta)_p$ is the specific heat at constant pressure, and $\alpha = -\frac{1}{\tau} \left(\frac{\partial \tau}{\partial \theta} \right)_p$

is the isobaric coefficient of thermal expansion. This leads to the relation

$$\theta_1 = (\alpha c_0 \theta_0 / c_p) u_0 \cos(kx - \omega t).$$

(b). The rate of production of entropy in a Newtonian viscous fluid is

$$\frac{\dot{h}}{\theta} + \frac{\kappa}{\theta^2} |\nabla \theta|^2 + \frac{\gamma}{\theta} (\text{tr} \underline{\underline{\varepsilon}})^2 + \frac{2\eta}{\theta} \text{tr} (\underline{\underline{D}} \cdot \underline{\underline{D}}).$$

Call this quantity P . Then the rate of dissipation of mechanical energy is

$$\dot{E} = -\theta P. \quad \text{In our case, } h = 0.$$

Neglecting terms higher than quadratic in the perturbed quantities, we obtain

$$\dot{E} = - \left[\frac{\kappa}{\theta_0} |\nabla \theta_1|^2 + \gamma (\text{tr} \underline{\underline{\varepsilon}})^2 + 2\eta (\underline{\underline{D}} : \underline{\underline{D}}) \right].$$

$$\text{Now } (\text{tr} \underline{\underline{\varepsilon}})^2 = (\partial u / \partial x)^2 \quad \text{and}$$

$$\underline{\underline{D}} : \underline{\underline{D}} = \frac{2}{3} (\partial u / \partial x)^2.$$

We thus find

$$\dot{E} = - \left[\kappa \theta_0 (\alpha c_0 / c_p)^2 + \left(\gamma + \frac{4}{3} \eta \right) \right] k^2 u_0^2 \sin^2(kx - \omega t).$$

The time average of $\sin^2(kx - \omega t)$ is $1/2$, so

$$\langle \dot{E} \rangle = -\frac{1}{2} k^2 u_0^2 \left[\left(\frac{4}{3} \eta + \gamma \right) + \kappa \theta_0 (\alpha c_0 / c_p)^2 \right].$$

Making use of the thermodynamic identity $c_p - c_v = \theta \alpha^2 c_0^2 c_v / c_p$, we obtain the desired result

$$\langle \dot{E} \rangle = -\frac{1}{2} k^2 u_0^2 \left[\left(\frac{4}{3} \eta + \gamma \right) + \kappa \left(\frac{1}{c_v} - \frac{1}{c_p} \right) \right].$$

To verify the identity, consider the

two equations of state $p = p(\tau, S)$ and $\theta = \theta(\tau, S)$. Differentiating these, let us define X, Y and Z by

$$dp = -X d\tau + Y dS$$

$$d\theta = -Y d\tau + Z dS.$$

The quantity $X = -(\partial p / \partial \tau)_S = \rho^2 (\partial p / \partial \rho)_S = \rho^2 c_0^2$ and $Z = (\partial \theta / \partial S)_\tau = \theta / c_v$. To find Y , we use the definition of c_p , viz.

$$c_p = \theta (\partial S / \partial \theta)_p.$$

When $dp = 0$, we have $d\tau = (Y/X) dS$ and therefore $d\theta = (1 - Y^2/XZ) Z dS$. Thus

$$c_p \text{ is given by } c_p = (1 - Y^2/XZ)^{-1} c_v \text{ or} \\ (*) Y^2 = (1 - c_v/c_p) X Z = (1 - c_p/c_v) \rho^2 c_0^2 \theta / c_v.$$

We may also write, when $dp = 0$,

$$d\theta = - (1 - XZ/Y^2) Y d\tau, \text{ and using the} \\ \text{definition } \alpha = -\frac{1}{\tau} (\partial \tau / \partial \theta)_p, \text{ we see that} \\ \text{we may also write } Y = (1 - XZ/Y^2)^{-1} \rho / \alpha \\ = (c_p/c_v - 1) \rho / \alpha (**).$$

Upon equating $*$ and $(**)^2$ we find the required identity $c_p - c_v = \theta^2 \alpha^2 c_0^2 c_v / c_p$.

- (c). The rate of decay of wave energy per unit distance will be $dE/dx = \langle \dot{E} \rangle / c_0$. In the absence of dissipation, the wave energy ~~density~~ density is $E = \frac{1}{2} \rho_0 u_0^2$. Since $c_0 = \omega/k$, we have $dE/dx = -E \{ (\omega^2 / \rho_0 c_0^3) [(\gamma + \frac{4}{3}\eta) + \kappa (c_v^{-1} - c_p^{-1})] \}$. Since $E \propto u_0^2$, the amplitude decay law will be $u = u_0 \exp \left\{ - (\omega^2 / 2 \rho_0 c_0^3) [(\frac{4}{3}\eta + \gamma) + \kappa (\frac{1}{c_v} - \frac{1}{c_p})] x \right\}$.

4. The best definition of Q (see O'Connell and Budiansky, GRL, 5, 5-8, 1978 on this fine point) is

$$Q^{-1} = \frac{\oint \dot{E} dt}{4\pi E_{ave}},$$

where $\oint \dot{E} dt$ is the net energy dissipated per cycle and E_{ave} is the average kinetic energy of oscillation. The latter, for the Slichter mode, is $E_{ave} = \langle \frac{1}{2} M \dot{s}^2 (1 + \beta \rho_1 / \rho_2) \rangle$ where M = mass of inner core and $\beta = \frac{1/2 + (a_2/a_1)^3}{1 - (a_2/a_1)^3} \approx 0.55$. The average of $\dot{s}^2$ is $\frac{1}{2} \dot{s}^2$, so

$$E_{ave} = \frac{1}{4} M \dot{s}^2 (1 + \beta \rho_1 / \rho_2).$$

The time-averaged energy dissipation above a 1 cm^2 plate moving with horizontal velocity $\dot{s} \cos \theta$ is $\frac{1}{2} \pi \rho_1 \delta (\dot{s} \cos \theta)^2$ erg/cm²-cycle where $\delta = (2\nu/\omega)^{1/2}$. Integrating this over the surface of the inner core gives

$$\begin{aligned} \oint \dot{E} dt &= 2\pi \int_0^\pi \frac{1}{2} \pi \rho_1 \delta \dot{s}^2 \cos^2 \theta \sin \theta d\theta \cdot a_2^2 \\ &= \frac{2}{3} \pi^2 \rho_1 \delta \dot{s}^2 a_2^2 \end{aligned}$$

Dividing, we get $Q^{-1} = \frac{\rho_1 / \rho_2}{1 + \beta \rho_1 / \rho_2} \left(\frac{\delta}{2a_2} \right)$.

With $\rho_1 \approx 11 \text{ gm/cm}^3$ and $\rho_2 = 13 \text{ gm/cm}^3$, $T = \frac{2\pi}{\omega} \approx 3.2 \text{ hrs}$ and $Q \approx 7 \cdot 10^6 \nu^{-1/2}$, where $[\nu] = \text{cm}^2/\text{sec}$. For $\nu_{\text{Gans}} = 10^{-2}$, $Q \approx 7 \cdot 10^7$ and for ν_{Toomre} , $Q \gtrsim 20,000$.