

## Problem set numero uno

1. The divergence of a vector field  $\underline{v} = v_i \hat{x}_i$  is  $\underline{\nabla} \cdot \underline{v} = \partial v_i / \partial x_i = \text{div } v_i$ . Show that this is the same in every Cartesian axis system, i.e. show that  $\underline{\nabla} \cdot \underline{v}$  is a geometrical object.

2. The Lagrangian description of a certain hump of jello is

$$r_1 = x_1 \cos \omega t + x_2 \sin \omega t$$

$$r_2 = -x_1 \sin \omega t + x_2 \cos \omega t$$

$$r_3 = x_3.$$

(a). Describe the motion geometrically. Find all particles which don't move. Draw a picture. Find  $\underline{u}_L(\underline{x}, t)$  and  $\underline{a}_L(\underline{x}, t)$ .

(b). Find  $\underline{u}_E(\underline{r}, t)$ , the Eulerian description of the motion.

3. The Eulerian description of a certain fluid motion is  $u_1 = r_1$ ,  $u_2 = -r_2$ ,  $u_3 = 0$ .

(a). Give the Lagrangian description. Find all particles which don't move. Describe the paths of the individual particles.



(b). Suppose the density of the fluid at  $t=0$  at the point  $x_1=1, x_2=1, x_3=0$  is  $\rho=7$ . Use the Lagrangian form of the law of conservation of mass to find  $\rho$  at this particle at time  $t=8$ . Show that this result could have been foreseen immediately by inspection of one of the Eulerian forms of the law of conservation of mass.

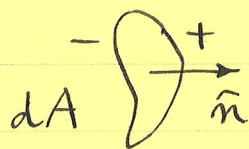
4. A streamline is a line in the continuum whose tangent is everywhere parallel to the instantaneous velocity  $\underline{u}(\underline{r}, t)$ . Under what circumstances do streamlines and particle paths coincide?

5. Let  $V(t)$  be a volume moving with some material,  $\phi$ -stuff, of density per unit volume  $\phi(\underline{r}, t)$ . Let  $\phi(\underline{r}, t) = \rho(\underline{r}, t) \psi(\underline{r}, t)$  so that  $\psi(\underline{r}, t)$  is the density of  $\phi$ -stuff per unit mass. Show that

$$\frac{d}{dt} \int_{V(t)} \rho \psi \, dV = \int_{V(t)} \rho D_t \psi \, dV.$$



6. A certain body of water undergoes a motion with Eulerian description  $\underline{u}(\underline{r}, t)$ . Dissolved in the water is a radioactive species with mean life  $1/\lambda$  (i.e. it decays like  $e^{-\lambda t}$ ). Because of concentration gradients, the flux of radioactive atoms relative to the water is  $\underline{F}$  gm cm<sup>-2</sup> sec<sup>-1</sup>. That is, if a small plane surface of area  $dA$  with unit normal  $\hat{n}$  moves with the water, then  $\hat{n} \cdot \underline{F} dA$  grams of the radioactive substance cross the surface from the - side to the + side every second. If  $\sigma$  is the



mass of the radioactive substance per cm<sup>3</sup> of space, find the appropriate generalization of the continuity equation for the radioactive substance. The equation relates  $\sigma$ ,  $\underline{F}$ ,  $\underline{u}$  and  $\lambda$ . Prove your result.

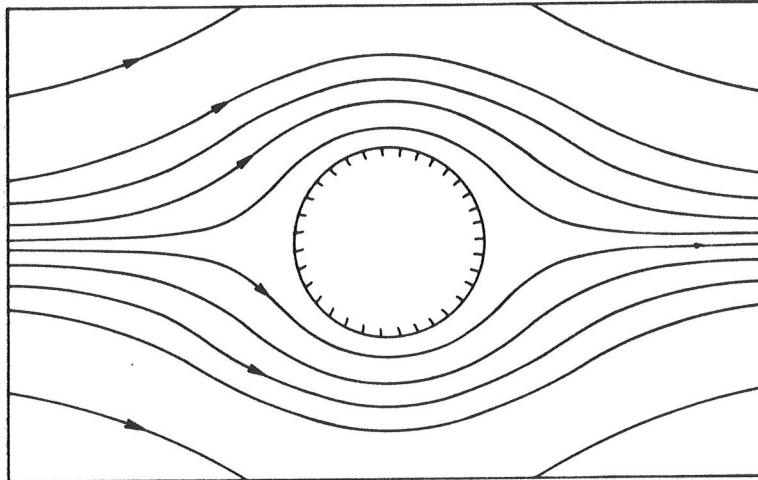


Figure 3.2 Low Reynolds number flow past a circular cylinder.

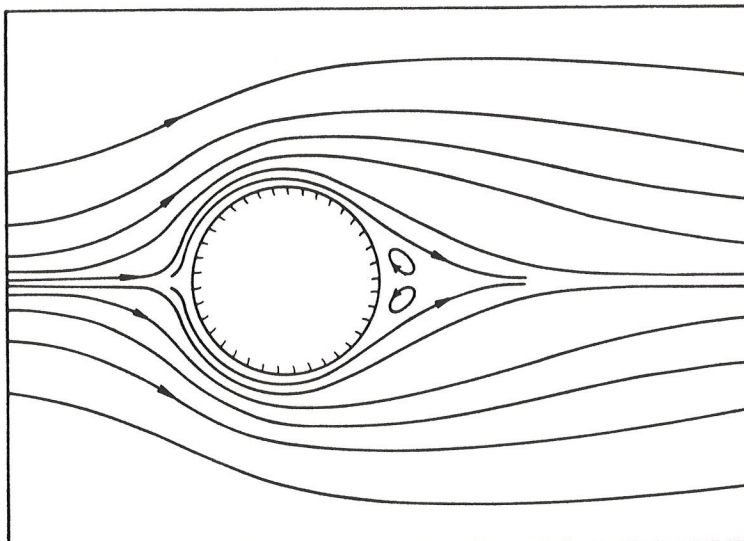


Figure 3.3 Flow past a circular cylinder at  $Re = 10$  (computed: Ref. [95]).



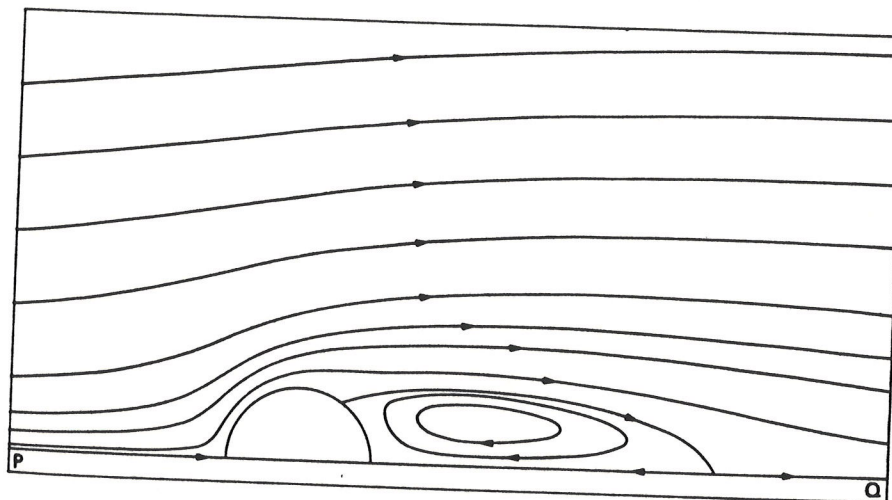


Figure 6.2 Flow past a circular cylinder at  $Re = 40$ . Ref. [95].

### *Flow Past a Circular Cylinder*

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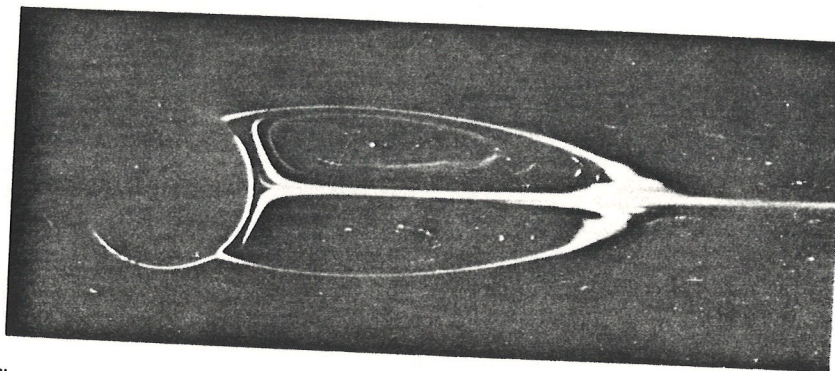


Figure 3.4 Attached eddies on circular cylinder at  $Re = 41$ , exhibited by coating cylinder with dye (condensed milk). From Ref. [245].

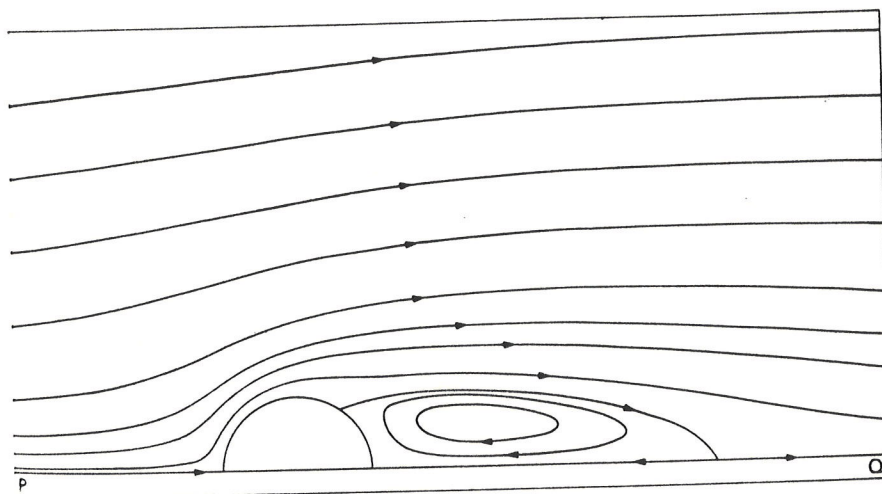


Figure 6.2 Flow past a circular cylinder at  $Re = 40$ . Ref. [95].

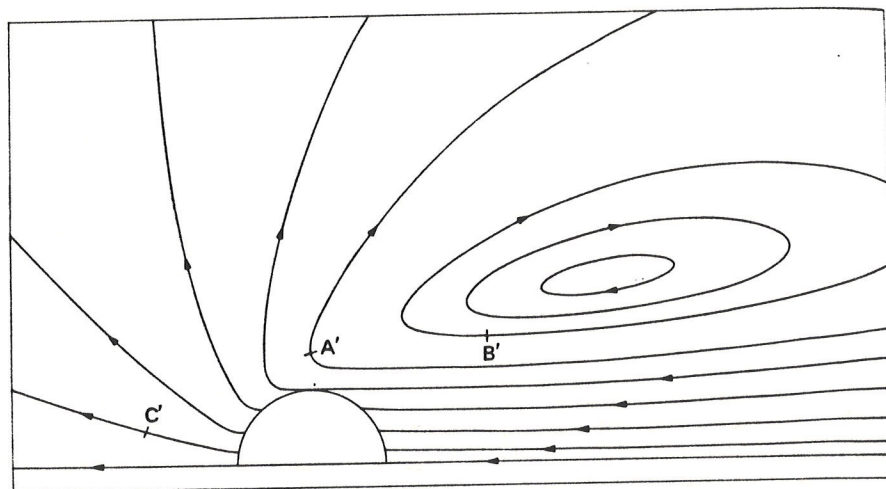


Figure 6.3 Streamlines for cylinder moving through stationary ambient fluid.



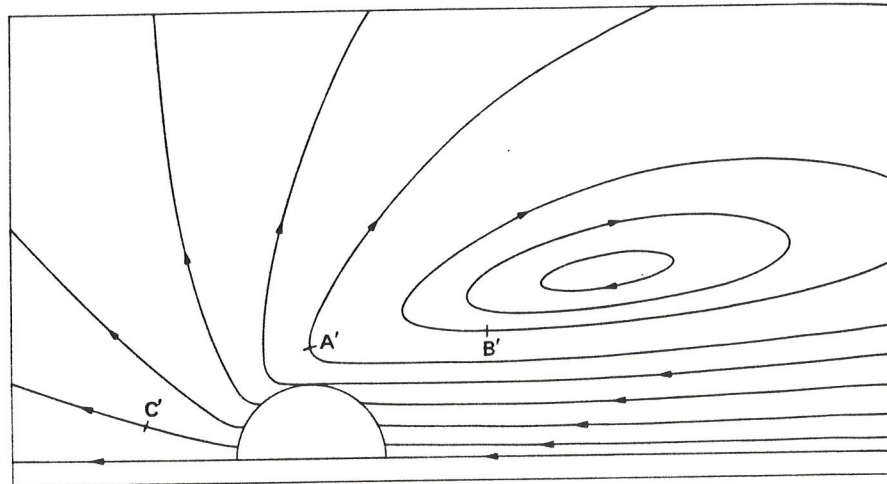


Figure 6.3 Streamlines for cylinder moving through stationary ambient fluid.

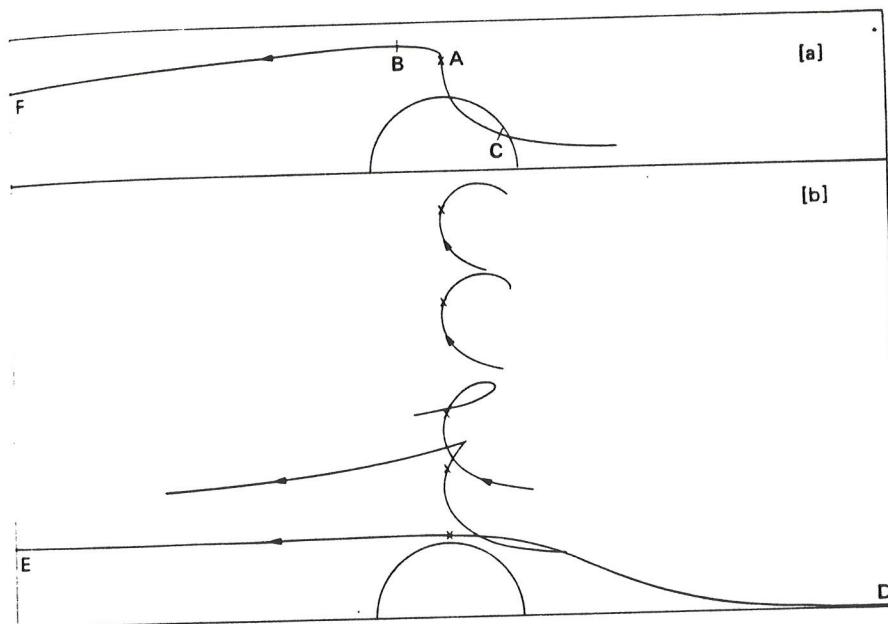


Figure 6.4 Particle paths for cylinder moving through stationary ambient fluid. See text for significance of parts (a) and (b).

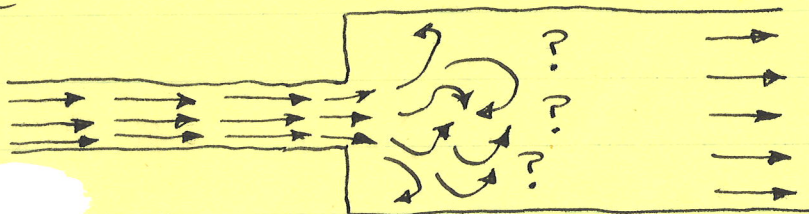
Problem set 2 : for those under stress,

1. In a certain material the components of the stress tensor are

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_3 & -r_2 \\ 0 & -r_2 & r_3 \end{bmatrix}.$$

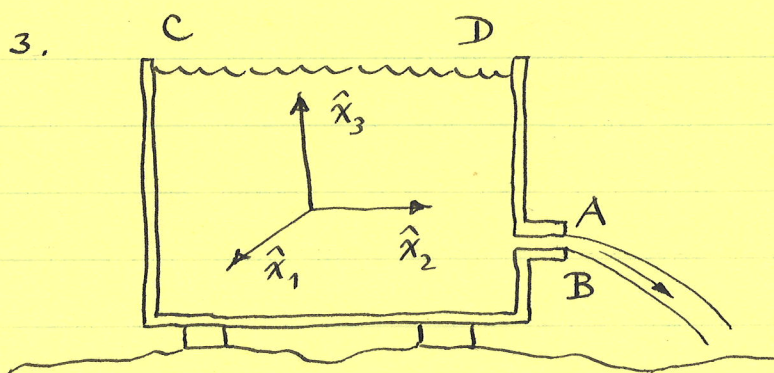
- (a). Find the traction at each point on the surface of the sphere  $r_1^2 + r_2^2 + r_3^2 = a^2$ . Find the torque per unit area about the origin exerted by this stress.
- (b). Find the total force exerted on the sphere's surface by the surrounding material, by evaluating a surface integral. In the same manner, find the total torque about the origin exerted on the sphere's surface.
- (c). Now find the force and torque in (b) by evaluating volume integrals instead of surface integrals.

2. Fluid is flowing in a pipe as shown





At a certain point, the cross-sectional area of the pipe changes suddenly from  $A_1$  to  $A_2$ . Everywhere upstream and far downstream of the change, the flow will be steady, i.e.  $\partial_t \underline{u} = \underline{0}$ . Neglect viscous drag on the pipe walls, and assume that ~~the flow is uniform~~ both upstream and far downstream, the flow is uniform across the pipe. If  $p_1$  and  $u_1$  are the pressure and velocity upstream, what are the pressure and velocity far downstream? Assume the density  $\rho$  of the fluid is constant.



A large tank of water, of constant density  $\rho$ , has a very small outlet pipe

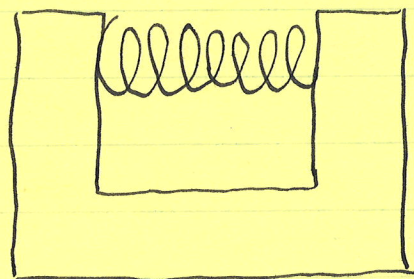
near the ~~bottom~~ bottom, from which the water empties out in a jet. If the tank is very large and the outlet pipe small,  $\partial_t \underline{u}$  will be zero to a good approximation everywhere in the tank and jet. Let  $a$  be the cross-sectional



area of the jet at AB. Assume the velocity of the water in the jet at AB is uniform,  $\underline{u} = U \hat{x}_2$ . Assume there is no atmosphere, so that the stress tensor in the water at AB and at the top CD vanishes.

- Use the momentum equation to find the total force exerted on the walls of the tank by the water.
- Compute the rate at which momentum leaks out of the tank in the jet.
- Why doesn't the momentum of the water in the tank change despite the leakage?

4.



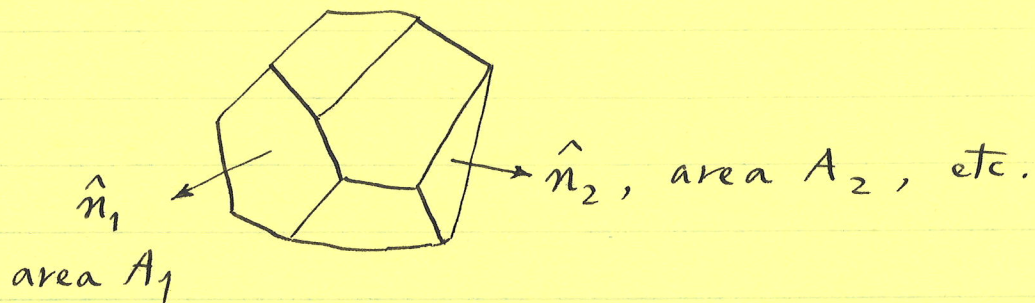
A large iron block has a slot cut in it as shown. A coil spring is compressed and placed in the slot, where it expands and pushes against the sides. Very roughly, sketch the flow lines in the block of the

$\hat{x}_2$  component of momentum.



5. Prove the polyhedron face theorem:  
 if a polyhedron has  $N$  faces  
 with areas  $A_i$ ,  $i = 1, 2, \dots, N$   
 and unit outward normals  
 $\hat{n}_i$ ,  $i = 1, 2, \dots, N$ , then

$$\sum_{i=1}^N A_i \hat{n}_i = \underline{0}.$$





# Problem set 2 1/2 : tensors are geometrical objects

1. What is the geometrical relationship between two vectors  $\underline{f}$  and  $\underline{g}$  if  $\underline{f}\underline{g} = \underline{g}\underline{f}$ ?

2. Find the traces of the following tensors  $T$ :

(a)  $T = \underline{f}\underline{g}$

(b)  $T = \underline{I}$  (identity tensor)

(c)  $T = A$  (alternating tensor; find  $\text{tr}_{12} T, \text{tr}_{13} T, \text{tr}_{23} T$ )

(d)  $T = \underline{f}\underline{g}\underline{h}$

(e)  ~~$T(\underline{u}, \underline{v})$~~   $T(\underline{u}, \underline{v}) = \underline{u} \cdot \underline{\Omega} \times \underline{v}$ ,  $\underline{\Omega}$  fixed vector,

(f)  $T(\underline{u}, \underline{v}, \underline{w}, \underline{x}) = (\underline{u} \cdot \underline{v})(\underline{w} \cdot \underline{x})$ . Find all traces.

3. An isotropic tensor is one whose components are the same in every Cartesian axis system. Every tensor of order zero is isotropic.

(a) Show that the only isotropic tensor of order 1 is the zero vector.

(b) Show that every isotropic tensor of order 2 is a multiple of  $\underline{I}$ .

(c) Show that the only isotropic tensor of order 3 is the zero tensor.



(d) Show that every isotropic tensor of order 4 is of the form  

$$C_{ijkl} = a \delta_{ij} \delta_{kl} + b \delta_{ik} \delta_{jl} + c \delta_{il} \delta_{jk}$$
 where  $a, b, c$  are constants.

4. Find the linear operator corresponding to the following tensors:

(a)  $T(\underline{u}, \underline{v}) = \underline{u} \cdot \underline{v}$

(b)  $T(\underline{u}, \underline{v}) = \underline{u} \cdot (\underline{\Omega} \times \underline{v})$ ,  $\underline{\Omega}$  fixed

(c)  $T = \underline{f} \underline{g}$

(d)  $T = T_{ij} \hat{x}_i \hat{x}_j$

5. Find the transposes of the linear operators in problem 4.

6. A certain linear operator  $Q$  has the following effect on a CAS  $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ :  
 $Q(\hat{x}_1) = \hat{x}_2 + \hat{x}_3$ ,  $Q(\hat{x}_2) = 3\hat{x}_1 + \hat{x}_2$ ,  $Q(\hat{x}_3) = 0$ .

(a) Find the components  $Q_{ij}$  of  $Q$  in the system  $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ .

(b) Find the components  $Q'_{ij}$  in the system  $\{\hat{x}'_1, \hat{x}'_2, \hat{x}'_3\}$  if  $\hat{x}'_1 = \frac{3}{5}\hat{x}_1 + \frac{4}{5}\hat{x}_3$ ,  $\hat{x}'_2 = \hat{x}_2$ ,  $\hat{x}'_3 = -\frac{4}{5}\hat{x}_1 + \frac{3}{5}\hat{x}_3$ .

(c) If  $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$  is left-handed, what is the handedness of  $\{\hat{x}'_1, \hat{x}'_2, \hat{x}'_3\}$ .

7. Use Gauss' theorem for vectors to prove Gauss' theorem for tensors  $T$  of order



q, namely 
$$\int_V \underline{r} \cdot \underline{T} \, dV = \int_{\partial V} \hat{n} \cdot \underline{T} \, dA.$$

Note that  $\underline{r}$  is replaced by  $\hat{n}$  and that if  $q > 2$ ,  $\hat{n} \cdot \underline{T} \neq \underline{T} \cdot \hat{n}$ , in general.

8. Let  $\lambda$  be a real number, and let  $\hat{n}$  be a unit vector. Show that the linear operator  $\phi$  which generates the tensor  $\lambda \hat{n} \hat{n}$  is symmetric, i.e.  $\phi = \phi^T$ . Find the components  $\phi_{ij}$  in a system of axes  $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ .

9. If  $\phi$  and  $\psi$  are linear operators, show that  $(\phi\psi)^T = \psi^T \phi^T$ .



Problem set 3 : for the energetic.

1. If a permanent steel magnet with magnetization density  $\underline{M}$  per unit volume is placed in a uniform magnetic field  $\underline{B}$ , the body torque on the magnet is  $\underline{M} \times \underline{B}$  per unit volume. For technical reasons, it is difficult to make this torque much larger than  $\sim 10^6$  dyne-cm / cm<sup>3</sup>

( $\mu_0 \underline{M} \sim \frac{1}{10}$  weber / m<sup>2</sup>,  $\underline{B} \sim 1$  weber / m<sup>2</sup>).

If a body torque of  $10^6 \hat{x}_3$  dyne-cm / cm<sup>3</sup> acts on a steel magnet, and the steel's internal angular momentum density does not change with time, and if the couple stress in the steel vanishes, find  $T_{ij} - T_{ji}$  in bars. If  $T_{ij} + T_{ji} = 0$ , sketch the directions of the surface tractions acting on the faces of a small cube with edges parallel to  $\hat{x}_1, \hat{x}_2, \hat{x}_3$ .

2. Mechanical potential energy : In situations where the body force  $\underline{f}(\underline{r}, t)$  comes from a time-independent external potential, i.e.  $\underline{f}(\underline{r}, t) = -\rho(\underline{r}, t) \nabla \Psi(\underline{r})$ , with  $\partial_t \Psi = 0$ , it is sometimes convenient to introduce the concept of potential energy. The mechanical potential energy in a volume  $V(t)$  is defined to be



$$\Psi = \int_{V(t)} \rho \psi \, dV.$$

Find a conservation law for  $\phi$ -stuff consisting of kinetic + ~~kinetic~~ internal + mechanical potential energy, i.e.

$\phi = \rho (E + \Psi) = \rho (\frac{1}{2} u^2 + u + \Psi)$ . Find the flux density and the rate of creation of this  $\phi$ -stuff in space.

3. Find a conservation law for the kinetic energy density per unit mass  $K = \frac{1}{2} u^2$ . Find the current density and rate of creation of kinetic energy in space.

4. Look again at problem #3 from problem set 2 (the water tank). Make the following assumptions about the water:

(i)  $\underline{H} = \underline{0}$  (no heat flow; this will be so if the temperature is uniform).

(ii)  $\underline{h} = \underline{0}$  (no absorption of sunlight, no radioactive or chemical heating).

(iii)  $\underline{u}$  (internal energy) is the same throughout the water at all times.

(iv)  $\underline{T}_{ij} = -p \delta_{ij}$  (i.e. ignore viscous stresses)

(v)  $\rho$  is the same throughout the



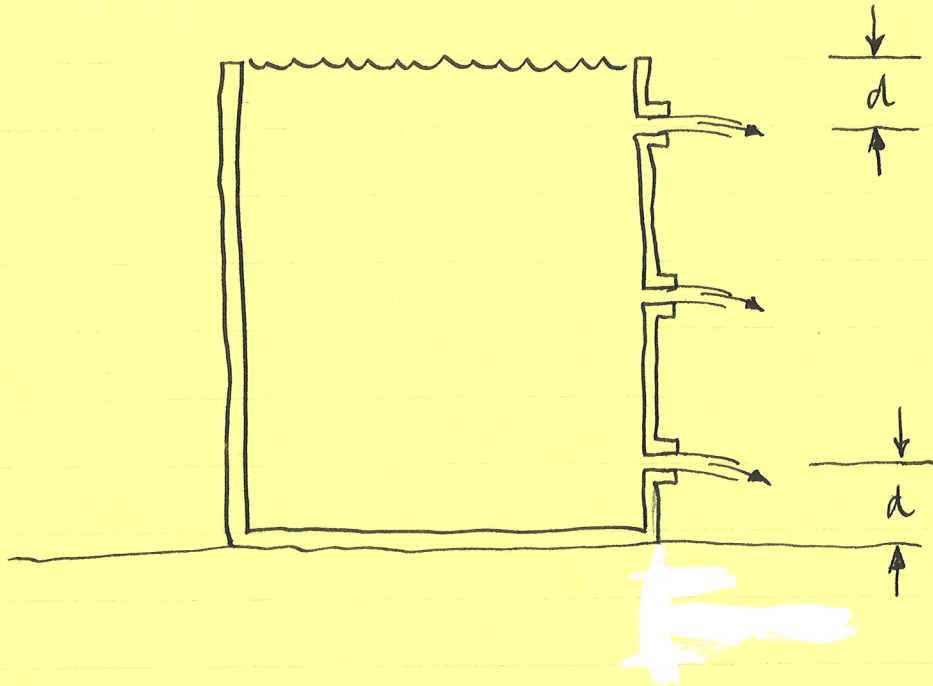
water at all times.

(vi) the acceleration of gravity,  $g$ , is constant.

To avoid confusion with the internal energy, re-name the outflow velocity in the jet, previously called  $u$ ; let's now call it  $V$ . Now:

- (a) Show that, with the above assumptions,  $\rho D_t u + \nabla \cdot \underline{H} = h + \nabla \cdot [\underline{T} \cdot \underline{v} u]$  is satisfied.
- (b) Find the current density of total energy  $E = K + u$  in the jet, and find the rate at which total energy is discharged from the pipe.
- (c) Find the rate at which the total energy of the water in the tank changes with time.
- (d) Account for the difference in (b) and (c), if any.
- (e) Repeat (b), (c) and (d) for the energy  $K + u + \Psi$ , where  $\Psi$  is gravitational potential energy.
- (f) Let  $H$  be the height of the water in the tank above the level of the jet. Find  $V$ , the outlet velocity in the jet.
- (g) Now consider a container with several outlet pipes as shown below.





Where should a pipe be placed in order that the water lands as far away on the table top as possible? Sketch the jet trajectories in the situation shown above. Perform a simple experiment to test your prediction.



Problem set #4: guaranteed to make you dizzy and gaseous.

1. The conservation laws we have derived are valid in an inertial reference frame (this was assumed implicitly in the derivation).

(a) Let  $\{\underline{0}; \hat{x}_1, \hat{x}_2, \hat{x}_3\}$  be an inertial frame with origin at  $\underline{0}$ , and let  $\{\underline{0}'; \hat{x}_1', \hat{x}_2', \hat{x}_3'\}$  be a frame which is rectilinearly accelerating w.r.t.  $\{\underline{0}; \hat{x}_1, \hat{x}_2, \hat{x}_3\}$  with an acceleration  $\underline{A}(t)$ , i.e. the relation between coordinates in the two frames is  $t' = t$ ,  $\underline{r}' = \underline{r} - \underline{R}(t)$  where  $\underline{A}(t) = d^2 \underline{R}(t)/dt^2$  (Galilean relativity). Show that in the accelerating frame, the conservation laws take the form

$$\begin{aligned} D_{t'}' \rho' + \rho' \nabla' \cdot \underline{u}' &= 0 \\ \rho' D_{t'}' \underline{u}' &= \nabla' \cdot \underline{T}' + (\underline{f}' - \rho' \underline{A}) \\ \rho' D_{t'}' \underline{U}' + \nabla' \cdot \underline{H}' &= \underline{T}' : \underline{\varepsilon}' + h' \end{aligned}$$

Be sure you can justify every step in the argument.

(b) Now let  $\{\underline{0}; \hat{x}_1', \hat{x}_2', \hat{x}_3'\}$  be a frame (note  $\underline{0}' = \underline{0}$ ) which is rotating relative to  $\{\underline{0}; \hat{x}_1, \hat{x}_2, \hat{x}_3\}$  with an instantaneous angular velocity  $\underline{\Omega}(t)$ . Show that in



the rotating frame

$$\begin{aligned}
 D_{t'}' \rho' + \nabla' \cdot \underline{u}' &= 0 \\
 \rho' [ D_{t'}' \underline{u}' + 2 \underline{\Omega} \times \underline{u}' + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}') + \frac{d' \underline{\Omega}}{d' t'} \times \underline{r}' ] \\
 &= \nabla' \cdot \underline{T}' + \underline{f}' \\
 \rho' D_{t'}' \underline{u}' + \nabla' \cdot \underline{H}' &= \underline{T}' : \underline{\underline{\epsilon}}' + \underline{h}'.
 \end{aligned}$$

Again, be careful to justify every step in the argument.

2. (Archimedes). A rigid body occupies a volume  $V$  in a certain fluid of constant density  $\rho$  at rest in a gravitational potential  $\phi$ . Show that the force and torque exerted on the body by the fluid are equal and opposite to the force and torque exerted by  $\underline{g} = -\nabla\phi$  on the displaced fluid. Now show how this result may be generalized to the case where the density  $\rho$  of the fluid is not constant.

Note : problems 3, 4 and 5 deal with ideal gases. You will need to know what they are.

3. Let  $\theta_0, S_0, p_0, \tau_0 = 1/\rho_0$  describe some particular equilibrium state of an ideal



gas whose ratio of specific heats is  $\gamma$ .

The gas may be a perfect fluid either because  $\theta = \theta_0$  throughout or because  $S = S_0$  throughout. Show that

(a) if  $\theta = \theta_0$  (isothermal gas), then the work function  $q$  may be taken as  $q = \tilde{R}\theta_0 \ln(p/p_0)$ , where  $\tilde{R} = R/M$ , with  $R$  the gas constant and  $M$  the molecular weight ( $\tilde{R}$  appears instead of  $R$  in continuum mechanics because it is customary to measure all extensive variables per gram rather than per mole).

(b) if  $S = S_0$  (isentropic gas), then  $q$  may be taken as  $q = \left(\frac{\gamma}{\gamma-1}\right) \tilde{R}\theta_0 (p/p_0)^{\frac{\gamma-1}{\gamma}}$ .

4. Let  $x, y, z$  be Cartesian coordinates.

Consider a gas in static equilibrium in the region  $z \geq 0$ , with  $\phi = g_0 z$  where  $g_0$  is a constant. Let  $\theta_0, S_0, p_0, \tau_0$  refer to the gas at  $z = 0$ .

(a) If  $\theta = \theta_0$  everywhere, show that  $p/p_0 = \rho/\rho_0 = e^{-z/h_0}$ , and find the "scale height"  $h_0$  in terms of  $\tilde{R}, \theta_0, g_0$ . What is  $h_0$  for ~~the~~ air at sea level if  $\theta_0 = 0^\circ\text{C} = 273^\circ\text{K}$ ?

(b) If  $S = S_0$  everywhere, find  $p/p_0$  and  $\rho/\rho_0$  as functions of  $\gamma, h_0$  and  $z$ . Show that  $p=0, \rho=0$  if  $z \geq ah_0$ , where  $a$  is a



finite number. Find  $a$  in terms of  $\delta$ .

5. Let  $r$  be radial distance from the center of a sphere of radius  $a$ . When  $r \geq a$ , suppose  $\phi = -g_0 a^2/r$ , where  $g_0$  is a constant. Suppose an ideal gas is in static equilibrium in the region  $r \geq a$ . Let  $\theta_0, S_0, p_0, \tau_0$  refer to the gas at  $r = a$ .

(a) If  $\theta = \theta_0$  everywhere, find  $p/p_0$  and  $\rho/\rho_0$  as functions of  $r$ . What is the total mass of the gas? How high is the level where  $p = 0$ ?

(b) If  $S = S_0$  everywhere, find  $p/p_0$  and  $\rho/\rho_0$  as functions of  $r$ . What is the total mass of the gas in this case? How high is the level where  $p = 0$ ?

Note: in problems 4 and 5, ~~the~~ self-gravitation, i.e. the contribution of the gas's mass to the gravitational field, is neglected. This is not a good approximation for large spherical masses of gas. For a discussion of the general case, see Chapter 4 of Chandrasekhar "An Introduction to the Study of Stellar Structure", Dover Publications.



6. (Sound propagation in a non-viscous fluid).

A fluid filling all of space and initially at rest has the following properties:

- (i) it is incapable of conducting heat, but is compressible.
- (ii) the stress tensor in the fluid is always  $\underline{T} = -p \underline{I}$ , where  $p$  is the thermodynamic equilibrium pressure.
- (iii) the temperature in the fluid may vary from place to place.
- (iv) no body forces act.

Now:

(a) Show that if the velocity of the fluid is zero (i.e. it is at rest), its pressure is constant, say  $p_0$ .

Note that its density may still vary from place to place if its temperature does.

(b) If this fluid is very slightly disturbed, its pressure will be  ~~$p_0$~~   $p_0 + p_1$  and its velocity will be  $\underline{u}$ , both  $p_1$  and  $\underline{u}$  being small. Show that if products of these small quantities are neglected,  $p_1$  satisfies an equation of the form

$$\frac{\partial^2 p_1}{\partial t^2} = c^2 \rho_0 \nabla \cdot (\rho_0^{-1} \nabla p_1), \quad *$$



where  $\rho_0$  is the density of the undisturbed fluid. Show how to calculate  $c$  from the thermodynamic properties of the fluid.

Hint #1: the relation of interest is

$$p = p(\rho, S).$$

Hint #2: This problem is not particularly easy. If you are having trouble, you might consult the brief but eloquent discussion of the thermodynamics of fluids in Chapter I of Eckart "Hydrodynamics of Oceans and Atmospheres".

(c) Evaluate  $c$  for air at  $0^\circ\text{C}$ .

(d) If  $\rho_0$  is constant and  $f$  is any function of a single variable, show that  $p_1 = f(x-ct)$  and  $p_1 = \frac{1}{r} f(r-ct)$  both satisfy  $\star$ . Here  $x, y, z$  are Cartesian coordinates, and  $r = (x^2 + y^2 + z^2)^{1/2}$ .



Problem set # 5 : dry water.

1. A perfect fluid of constant density  $\rho$  is at rest and fills all of space. No body forces act. Suddenly last summer (at time  $t = 0$ ), the fluid inside a spherical cavity of radius  $a$  vanishes. The resulting cavity immediately begins to collapse ("nature abhors a vacuum"). During the collapse the pressure within the cavity remains zero, and the pressure in the fluid at  $\infty$  remains  $P$ .

(a). Show that if  $R(t)$  is the radius of the bubble at time  $t$ , then at time  $t$  the velocity potential is  $\psi = -\dot{R}(t)R^2(t)/r$ , where  $r$  = radial distance from origin (at center of cavity).

(b). Show that  $\ddot{R}R + \frac{3}{2}(\dot{R})^2 + P/\rho = 0$  } do not  
 (c). Show that  $R^3(\dot{R})^2 = \frac{2}{3}(P/\rho)(a^3 - R^3)$  } use  
 Lagrangians.

(d). Find the pressure  $p$  in the fluid as a function of  $r$  when the cavity radius is  $R$ . Where is  $p$  largest, and what is its maximum value?

(e). How long does it take the cavity to collapse completely?

(f). The amount of work required to open a cavity of volume  $V$  against a



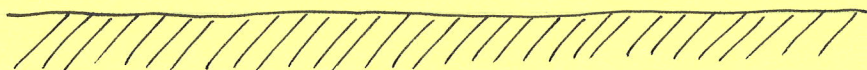
constant pressure  $P$  is  $PV$ . Regarding this work as stored potential energy, find the Lagrangian of the cavity-fluid system. From it, deduce equations (b) and (c).

2. A massless ping-pong ball is being held underwater in an ocean consisting of incompressible water of constant density. At time  $t=0$  it is released. Describe its subsequent motion. You may ignore the presence of the top surface of the water, the ocean bottom and the hand holding the ball (i.e. take the ocean to be of  $\infty$  extent). Do the problem two ways: (a) with Lagrangians and (b) without.

~~~~~



Note: hand  
not shown  
to scale.





3. Consider a self-gravitating sphere of radius  $a$ , consisting of an incompressible homogeneous fluid of constant density  $\rho$ . Its mass is  $M = \frac{4}{3} \pi G \rho a^3$ .

(a). Suppose that its shape is changed slightly so that the outer surface, originally at  $r=a$ , is displaced to  $r=a+\xi$  where  $\xi \ll a$ . Show that the resulting gravitational potential  $\phi$  on the deformed surface is given, to first order in  $\xi/a$ , by

$$\phi_{\text{surface}} = -\frac{GM}{a} \left[ 1 - \sum_l \sum_m \frac{2(l-1)}{2l+1} \left( \frac{\xi_l^m}{a} \right) Y_l^m \right],$$

where  $\xi = \sum_l \sum_m \xi_l^m Y_l^m$  is the spherical harmonic expansion of  $\xi$ .

(b) (Kelvin). If the sphere is so deformed, it will undergo infinitesimal free oscillations. Show that the eigenfrequencies  $\omega_l$  are given by

$$\omega_l^2 = \frac{2l(l-1)}{2l+1} \left[ \frac{4}{3} \pi G \rho \right].$$



- (c). Discuss the special cases  $l=0$  and  $l=1$ .
- (d). With  $\rho = 5.517 \text{ gm/cm}^3$  and  $a = 6371 \text{ km}$ , what is the period of the fundamental mode ( $l=2$ )?

4. (Capillary waves). (a) Consider a surface defined by  $z = h(x, y)$  where  $| \nabla h | \ll 1$ . Show that, correct to first order in  $| \nabla h |$ ,

$$\nabla \cdot \hat{n} = - \nabla^2 h$$

where  $\hat{n}$  is the unit normal to the surface.

(b). Show that, ~~when the surface tension of the air-water interface is taken into account, the dispersion relation for surface waves on water of depth  $H$  is modified to~~ when the surface tension of the air-water interface is taken into account, the dispersion relation for surface waves on water of depth  $H$  is modified to

$$\omega^2 = (gk + \gamma k^3 / \rho) \tanh kH$$

where  $g$  is the acceleration of gravity,  $\rho$  is the density and  $\gamma$  is the surface tension coefficient ( $\gamma = 72 \text{ dyne/cm}$  for air-water).

(c). For simplicity, restrict attention ~~now~~ now to the deep water case  $kH \gg 1$ . Draw a



plot of the phase velocity  $c = \omega/k$  vs. the wavelength  $\lambda = 2\pi/k$ . At what value of  $\lambda$  is  $c$  a minimum? What is the corresponding value of  $c$  at  $\lambda_{\min}$ ? What are  $\lambda_{\min}$  and  $c_{\min}$  for  $H_2O$ ?

(d). Now plot the group velocity  $u = d\omega/dk$  vs.  $\lambda$ . At what value of  $\lambda$  is  $u$  a minimum? The group velocity has the following property: if waves are excited by a point source at  $r=0$  ( $r, \theta, z$  being cylindrical coordinates) and  $t=0$  (i.e. if a stone is thrown into the water), then some time later waves of wavelength  $\lambda$  will be found at a distance  $r = u(\lambda)t$  from the source. Since  $u$  has a minimum,  $u_{\min}$ , when a stone is thrown into the water, a calm region should appear in the middle, surrounded by waves travelling at  $u_{\min}$ . For water, how rapidly should the calm region spread (i.e. what is  $u_{\min}$ ) and what is the wavelength of the waves just outside the region?

(e). Now go down to Lake Carnegie and test the predictions of your theory.



## Sticky problems # 6

1. An incompressible Newtonian viscous fluid of constant density  $\rho$  and viscosity  $\eta$  flows down an incline of slope  $\alpha$  in a gravitational field  $-g\hat{z}'$ . Assume the flow is laminar and that the upper surface of the fluid is free, in contact with air whose stress tensor is  $-p_A \underline{\underline{I}}$ , where  $p_A$  is constant. Find:

- the velocity profile as a function of depth in the fluid
- the total stress throughout the fluid
- the stress exerted on the incline by the fluid
- the heat generated per  $\text{cm}^3$  per sec by viscous friction in the fluid.
- the temperature in the fluid, assuming that both top and bottom are at  $20^\circ\text{C}$ , and that the thermal conductivity  $\kappa$  is constant.

2. A pail, which once belonged to Sir Isaac Newton, is filled with water. At  $t=0$ , one begins to rotate the pail with a steady



angular velocity. At first, the water does not rotate, but after a while it is observed to be rotating rigidly with the pail. For simplicity, consider an infinitely long rigid cylinder ~~of~~ of radius  $a$  filled with fluid of density  $\rho$  and viscosity  $\eta$ . The gravitational field is  $-g\hat{z}$ . At  $t=0$ , the cylinder begins to rotate with a uniform angular velocity ~~of~~  $\Omega$  about the  $\hat{z}$  axis.

- (a). Assuming the flow is laminar, determine the motion of the enclosed fluid.
- (b). How does the "boundary layer thickness" grow with time?
- (c). How long does it take for the fluid to be essentially co-rotating with the container.
- (d). Let  $a$  be the radius of a coffee cup or beaker, and let the fluid be water (or coffee). What is the answer in seconds?
- (e). Do a simple experiment to observe this phenomenon. Hint: spin-down is easier to observe than spin-up. Just note how long it takes the coffee to come to rest after it has been



stirred into a state approximating rigid rotation. Does your theory agree with observation? Discuss.

3. (The attenuation of sound waves).

A fluid of constant density  $\rho_0$ , constant pressure  $p_0$ , constant temperature  $\theta_0$  and constant sound speed  $c_0$  fills all of space.

(a). Show that if a propagating infinitesimal sound wave has a velocity field of the form

$$\underline{u} = u_0 \hat{x} \cos(kx - \omega t),$$

the associated density, pressure and temperature fluctuations are given by

$$\rho_1 = (\rho_0/c_0) u_0 \cos(kx - \omega t)$$

$$p_1 = (\rho_0 c_0) u_0 \cos(kx - \omega t)$$

$$\theta_1 = (\alpha c_0 \theta_0 / c_p) u_0 \cos(kx - \omega t),$$

where  $\alpha$  is the coefficient of thermal expansion and  $c_p$  is the specific heat at constant pressure.

(b). Using the above expressions, calculate the time-averaged rate of energy dissipation (per  $\text{cm}^3$  per sec)  $\langle \dot{E} \rangle$  in the fluid, due to both thermal conduction and viscosity. Show that

$$\langle \dot{E} \rangle = -\frac{1}{2} k^2 u_0^2 \left[ \left( \frac{4}{3} \eta + \gamma \right) + \kappa (c_v^{-1} - c_p^{-1}) \right],$$

where  $\eta$  and  $\gamma$  are the shear and bulk



viscosities,  $\kappa$  is the thermal conductivity and  $c_v$  is the specific heat at constant volume. You will need to use, and should verify, the thermodynamic identity

$$c_p - c_v = \theta \alpha^2 c_0^2 c_v / c_p.$$

(c). Show that the amplitude of the sound wave will be attenuated by viscosity and heat conduction by the amount  $u_0 = u_0 e^{-\beta x}$ , where the absorption coefficient  $\beta$  is given by

$$\beta = \frac{\omega^2}{2\rho_0 c_0^3} \left[ \left( \frac{4}{3}\eta + \gamma \right) + \kappa (c_v^{-1} - c_p^{-1}) \right].$$

The above method, which calculates  $\langle \dot{E} \rangle$  and  $\beta$  using the dissipation-free fields  $u$ ,  $p_1$ ,  $\dot{p}_1$  and  $\theta_1$ , will be valid as long as the dissipation per wavelength is small, i.e. provided  $\beta k^{-1} \ll 1$ . For ~~the case of~~ common fluids at ordinary pressures and temperatures, this is demonstrably so.

4. Estimate the  $Q$  of the Slichter mode assuming that all the dissipation takes place in a thin viscous boundary layer surrounding the inner core ( $2\pi Q^{-1} \equiv$  the fractional energy dissipated per cycle). Obtain a



numerical estimate for the two values  $\nu = 10^{-2} \text{ cm}^2/\text{s}$  obtained by Gans, J. G. R., 77, 360-66 (1972), based on solid-state arguments and  $\nu < 10^5 \text{ cm}^2/\text{s}$  obtained by Toomre, GJRAS, 38, 335-348 (1974) on the basis of ~~the~~ the lack of an observed phase lag of the 18.6 year nutation (the latter is a firm bound; the former is necessarily more speculative but has come to be widely accepted).